What about the Helmholtz E_φ equation?

This stuff is now incorporated as Appendix section D.3.

Here are the E_r and E_φ equations in m space:

$[\nabla^2\mathbf{E}]_r + \beta^2 E_r = 0$:

$$[r^2\partial_r^2 + r\partial_r - (m^2+1) + r^2(\beta^2-\beta_d^2)]\, E_r(r,m) - 2jm\, E_\varphi(r,m) = 0 \qquad \text{(D.1.17)}$$

$[\nabla^2\mathbf{E}]_\varphi + \beta^2 E_\varphi = 0$:

$$[r^2\partial_r^2 + r\partial_r - (m^2+1) + r^2(\beta^2-\beta_d^2)]\, E_\varphi(r,m) + 2jmE_r(r,m) = 0 \qquad \text{(D.1.18)}$$

How do I know that my solutions E_φ and E_r satisfy this last equation? Maybe they don't! Why should they?

Rewrite the above in this manner

$$[x^2\partial_x^2 + x\partial_x - (m^2+1) + x^2]\, E_r(r,m) - 2m\, jE_\varphi(r,m) = 0 \qquad \text{(D.1.17)}$$

$$[x^2\partial_x^2 + x\partial_x - (m^2+1) + x^2]\, jE_\varphi(r,m) - 2mE_r(r,m) = 0 \qquad \text{(D.1.18)}$$

I solved the first equation for E_φ in my work, and I solved the H_z equation for E_r. So what happens when I insert these two solutions into the last equation above? Rewrite the last as

$$[x^2\partial_x^2 + x\partial_x - (m^2+1) + x^2]\, jE_\varphi(r,m) = 2m\, E_r(r,m).$$

It would certainly be nice if this were true. I have

$$E_r(r,m) = a_m\, x^{-1}\, J_m(x) + \frac{K_m}{2}\, J_{m+1}(x) \quad . \qquad \textit{(D.2.11)}$$

$$jE_\varphi(r,m) = -\, a_m\, x^{-1}\, J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x) \qquad x = \beta' r \quad . \qquad \text{(D.2.15)}$$

Then we get

$$\text{LHS} = [x^2\partial_x^2 + x\partial_x - (m^2+1) + x^2]\, \{-\, a_m\, x^{-1}\, J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x)\, \}$$

$$\text{RHS} = 2m\{\, a_m\, x^{-1}\, J_m(x) + \frac{K_m}{2}\, J_{m+1}(x)\}$$

You would think this has to balance separately as if K_m and a_m were independent constants. So then

$$\text{LHS}_K = [x^2\partial_x^2 + x\partial_x - (m^2+1) + x^2]\, \left(\frac{K_m}{2} J_{m+1}(x)\right)$$

$$RHS_K = 2m \frac{K_m}{2} J_{m+1}(x)$$

Then I would have to show that

$$[x^2\partial_x^2 + x\partial_x - (m^2+1) + x^2] J_{m+1}(x) = 2m J_{m+1}(x)$$

or

$$[x^2\partial_x^2 + x\partial_x - (m^2+1) + x^2 - 2m] J_{m+1}(x) = 0$$

or

$$[x^2\partial_x^2 + x\partial_x - (m^2+1+2m) + x^2] J_{m+1}(x) = 0$$

or

$$[x^2\partial_x^2 + x\partial_x - (m+1)^2 + x^2] J_{m+1}(x) = 0$$

or

$$[x^2\partial_x^2 + x\partial_x + x^2 - (m+1)^2] J_{m+1}(x) = 0$$

Bessel's equation does say that

$$[x^2\partial_x^2 + x\partial_x + x^2 - (m+1)^2] J_{m+1}(x) = 0$$

so it works!!!!! I am totally amazed. At least the K_m terms balance.

Now let's try the a_m balance.

$$LHS = [x^2\partial_x^2 + x\partial_x - (m^2+1) + x^2] \{- a_m x^{-1} J_m(x) + \frac{a_m}{m} J_{m+1}(x) \}$$
$$RHS = 2m\{ a_m x^{-1} J_m(x) \}$$

$$LHS = [x^2\partial_x^2 + x\partial_x - (m^2+1) + x^2] \{- x^{-1} J_m(x) + m^{-1} J_{m+1}(x) \}$$
$$RHS = 2m\{ x^{-1} J_m(x) \}$$

This is a much messier situation. I could try Maple. Here is Maple evaluating the LHS:

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LHS := x^2*diff(f(x),x,x)+ x*diff(f(x),x) - (m^2+1)*f(x) + x^2*f(x);
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$$LHS := x^2\left(\frac{\partial^2}{\partial x^2} f(x)\right) + x\left(\frac{\partial}{\partial x} f(x)\right) - (m^2+1) f(x) + x^2 f(x)$$

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f := x -> -(1/x)*BesselJ(m,x) + (1/m)*BesselJ(m+1,x);
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$$f := x \to -\frac{\text{BesselJ}(m, x)}{x} + \frac{\text{BesselJ}(m+1, x)}{m}$$

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LHS := x^2*diff(f(x),x,x)+ x*diff(f(x),x) - (m^2+1)*f(x) + x^2*f(x): simplify(%);
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$$2\frac{m\,\text{BesselJ}(m, x)}{x}$$

But this is exactly the RHS !! **I am again astounded to see this working right.**

Now I can consider the notion of trying to find the m = 0 solution for E_φ using the φ equation,

$$[r^2\partial_r^2 + r\partial_r - (m^2+1) + r^2\beta'^2]\, E_\varphi(r,m) + 2jmE_r(r,m) = 0$$

which for m = 0 becomes

$$[r^2\partial_r^2 + r\partial_r - 1 + r^2\beta'^2]\, E_\varphi(r,m=0) = 0$$

or

$$[x^2\partial_x^2 + x\partial_x - 1 + x^2]\, E_\varphi(r,m=0) = 0$$

This seems to say that

$$E_\varphi(r,0) = C_{\varphi 0}\ J_1(x)$$

where $C_{\varphi 0}$ is some constant.

How does this then figure into my boundary conditions stuff? If I want this to be zero on the surface using my equipotential idea, then for m =0 I must conclude that $C_{\varphi 0} = 0$. But that is not required, and I suppose you could allow an m = 0 rotating current on the surface?

Question: Can the φ Helm be derived from the others and div E = 0?

$[\nabla^2\mathbf{E}]_z + \beta^2 E_z = 0$:

$$[r^2\partial_r^2 + r\,\partial_r - m^2 + r^2\,\beta'^2]\, E_z(r,m) = 0 \qquad \text{ok} \qquad \text{(D.1.16)}$$

$[\nabla^2\mathbf{E}]_r + \beta^2 E_r = 0$:

$$[r^2\partial_r^2 + r\partial_r - (m^2+1) + r^2\,\beta'^2]\, E_r(r,m) - 2jm\, E_\varphi(r,m) = 0 \qquad \text{(D.1.17)}$$

$[\nabla^2\mathbf{E}]_\varphi + \beta^2 E_\varphi = 0$:

$$[r^2\partial_r^2 + r\partial_r - (m^2+1) + r^2\beta'^2]\, E_\varphi(r,m) + 2jmE_r(r,m) = 0 \qquad \text{(D.1.18)}$$

div **E** = 0 :

$$\partial_r\,[r\,E_r(r,m)] + jmE_\varphi(r,m) - j\beta_d\, r\, E_z(r,m) = 0 \qquad \text{(D.1.19)}$$

Maybe ponder in Cartesians first.

$$(\partial_j^2 + \beta^2)\, E_i = 0$$

$$\partial_j E_j = 0$$

$$(\partial_1^2 + \partial_2^2 + \partial_3^2 + \beta^2)\, E_1 = 0$$

$$(\partial_1^2 + \partial_2^2 + \partial_3^2 + \beta^2)\, E_2 = 0$$

$(\partial_1^2+\partial_2^2+\partial_3^2+\beta^2)\,E_3 = ????$

Start with

$\partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3 = 0$

then

$(\partial_1^2+\partial_2^2+\partial_3^2+\beta^2)\partial_1 E_1 + (\partial_1^2+\partial_2^2+\partial_3^2+\beta^2)\partial_2 E_2 + (\partial_1^2+\partial_2^2+\partial_3^2+\beta^2)\partial_3 E_3 = 0$

or

$\partial_1 (\partial_1^2+\partial_2^2+\partial_3^2+\beta^2)\,E_1 + \partial_2 (\partial_1^2+\partial_2^2+\partial_3^2+\beta^2)\,E_2 + \partial_3 (\partial_1^2+\partial_2^2+\partial_3^2+\beta^2)\,E_3 = 0$

or

$\partial_1\, 0 + \partial_2\, 0 + \partial_3 (\partial_1^2+\partial_2^2+\partial_3^2+\beta^2)\,E_3 = 0$

or

$\partial_3 [(\partial_1^2+\partial_2^2+\partial_3^2+\beta^2)\,E_3] = 0$

Well it certainly works but not a hard proof. Go back to

$(\nabla^2+\beta^2)\mathbf{E} = 0 \quad \Rightarrow \quad \text{grad}(\text{div}\ \mathbf{E}) - \text{curl}\ (\text{curl}\ \mathbf{E}) + \beta^2\, E = 0$

When div E = 0 this says

$-\ \text{curl}\ (\text{curl}\ \mathbf{E}) + \beta^2\, \mathbf{E} = 0$