App D Paradox of the Day 10.5.14 PhL 10.5.14

Retrograde motion on Appendix D.9 ! I had it right long ago, then I changed it , and now I change it back. It is a tricky notational issue really, k → k' called k, I think the new D.9 clarifies this tricky issue and I did several tests to make sure it is now right.

1. On the one hand, I have argued in Appendix D.9 that when we change from G = 0 to G > 0, the Appendix D solutions change simply by the addition of the factor (ξ_2/ε_2) to all the E and B fields. For example, when G = 0 the E field solutions are,

Second summary of the E field solutions : (D.2.33)

$$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\ (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\ (ak)\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

$$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\ (ak)\ h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

where

$$I = 2\pi a\,(\omega/k)\,N_0 = 2\pi a\,(\omega/k)\,(1/2\pi a)\,q = (\omega/k)\,CV \quad . \qquad (D.2.31c)$$

Thus, for G = 0 the E fields are

$$E_z(r,m) = (1/4)\,\eta_m\ (\omega/k)\,CV\,R_{dc}\ (a\beta')\, f_m$$
$$E_r(r,m) = (j/4)\,\eta_m\ (\omega/k)\,CV\ R_{dc}\ (ak)\, g_m$$
$$E_\theta(r,m) = (1/4)\,\eta_m\ (\omega/k)\,CV\ R_{dc}\ (ak)\ h_m$$

and for G > 0 the fields are

$$E_z(r,m) = (\xi_2/\varepsilon_2)\,(1/4)\,\eta_m\ (\omega/k)\,CV\,R_{dc}\ (a\beta')\, f_m$$
$$E_r(r,m) = (\xi_2/\varepsilon_2)\,(j/4)\,\eta_m\ (\omega/k)\,CV\ R_{dc}\ (ak)\, g_m$$
$$E_\theta(r,m) = (\xi_2/\varepsilon_2)\,(1/4)\,\eta_m\ (\omega/k)\,CV\ R_{dc}\ (ak)\ h_m$$

The entire change can be encapsulated by saying

$$I = (\omega/k)\,CV \qquad \rightarrow \qquad I = (\omega/k)\,CV(\xi_2/\varepsilon_2)$$

which we can write as

$$I_{G>0} / I_{G=0} = (\xi_2/\varepsilon_2)$$

2. On the other hand, I know that

$$1/Z_0 = \sqrt{\frac{G + j\omega C}{R + j\omega L}}$$

where for low ω we have $R = R_{dc2}$. I also know that

$$I = V/Z_0$$

Therefore

$$I = V \sqrt{\frac{G + j\omega C}{R + j\omega L}} \qquad G > 0$$

For G = 0 this reads

$$I = V \sqrt{\frac{j\omega C}{R + j\omega L}} \qquad G = 0$$

The ratio by which I increases is this:

$$I_{G>0} / I_{G=0} = \sqrt{\frac{G + j\omega C}{R + j\omega L}} \Big/ \sqrt{\frac{j\omega C}{R + j\omega L}} = \sqrt{\frac{G+j\omega C}{j\omega C}}$$

3. The Paradox: Comparing these two results, we must have

$$(\xi_2/\varepsilon_2) = \sqrt{\frac{G+j\omega C}{j\omega C}}$$

But

$$(\xi_d/\varepsilon_d) = (\varepsilon_d - j\sigma_d/\omega)/\varepsilon_d = 1 - j\,(\sigma_d/\varepsilon_d)\,(1/\omega) = 1 - j\,(G/C)\,(1/\omega) = 1 + G/(j\omega C)$$

Thus, it must be true that

$$1 + G/(j\omega C) = \sqrt{\frac{G+j\omega C}{j\omega C}}$$

or

$$\frac{G+j\omega C}{j\omega C} = \sqrt{\frac{G+j\omega C}{j\omega C}}$$

So THAT is the paradox.

4. Think source is in Appendix D.9 (d)

I do remember dealing with this square root issue once before in Appendix D.9, then I got rid of the square root. So I guess to resolve this I need to do yet another review of that section. I see a comment in red on this in "old App D with betad.doc".

Plan A [ignore this, go to Plan B **]**

At large ω, we know that that for G = 0, $k^2 = \beta_{d0}^2 = \omega^2 \mu_d \varepsilon_d$.
But when G > 0, we have $k^2 = \beta_d^2 = \omega^2 \mu_d \xi_d$. [wrong!]
Are these claims true, or not?

The values for k^2 come from the usual k(ω) formula in the large ω limit, as for example in (Q.3.5),

$$\text{Re}(k) \approx (\omega/v_d) \qquad v_d = 1/\sqrt{L_e C} \approx 1/\sqrt{LC}$$
$$\text{Im}(k) \approx -\omega \tan_L /(2v_d)$$

where it says $v_d = 1/\sqrt{L_e C} \approx 1/\sqrt{LC}$. Why does this not depend on whether $G_{dc} = 0$ or $G_{dc} > 0$? Well, the model of (Q.1.9) says

$$G(\omega) = C(\omega_d + \tan_L \omega) \qquad \omega_d \equiv (\sigma_d/\varepsilon_d) \qquad G_{dc} \equiv \omega_d C$$

where $\omega_d = G_{dc}/C$. But when ω is high enough, you ignore ω_d and so the result is independent of G_{dc}. That is the reason.

Meanwhile,

$$L_e C = \mu_d \varepsilon_d = 1/v_d^2 \qquad (4.1219)$$

It seems that G_{dc} has no effect whatsoever on the value of k for large ω.

$$\beta_{d0}^2 = \omega^2 \mu_d \varepsilon_d = \omega^2/v_d^2 \qquad \beta_{d0} = \omega/v_d$$

so only β_{d0} is involved at large ω. If you "turn on" the dielectric conductivity at high ω, there is no change in the value of k (assuming you have not altered $\tan_L$ in this turn on).

OK, for high ω this is the case.

Plan B.

Start here:

$$N_m \rightarrow (\xi_d/\varepsilon_d) N_m \quad \text{everywhere .} \qquad (D.9.23)$$

For example, for the total current we have from (D.2.31a) that $I = 2\pi a\ (\omega/k)\ N_0$, so here is the change in symbol I caused by turning on the conducting dielectric,

$$I = 2\pi a\ \{N_0\}\ \frac{\omega}{k(G_{dc}=0)}\ \rightarrow\ 2\pi a\ \{(\xi_d/\varepsilon_d)N_0\}\ \frac{\omega}{k(G_{dc}>0)}\ .$$

We must now treat a subtle point regarding k in order to obtain the correct "rule" for how turning on the dielectric DC conductivity alters the E and B fields presented earlier in Appendix D. Although k is treated as a generic constant in Appendix D, we will eventually be setting it to the specific value k(ω),

$$k = k(\omega) \equiv -j\sqrt{\mathbf{zy}}\ = -j\sqrt{(R(\omega)+j\omega L(\omega))(G(\omega)+j\omega C(\omega))} \qquad (5.3.6)$$

where the four parameters are as given in the simple model of (Q.1.9). In particular, C(ω) = C, a constant, whereas $G(\omega) = G_{dc} + C\tan_L\omega$. so we may write

$$k = -j\sqrt{(R(\omega)+j\omega L(\omega))(\ G_{dc} + C\tan_L\omega + j\omega C)}\ \approx\ -j\sqrt{(R(\omega)+j\omega L(\omega))(\ G_{dc} + j\omega C)}$$

which shows the traditional dependence on G_{dc} and C. Thus we can write

$$k(G_{dc}> 0) =\ -j\sqrt{(R(\omega)+j\omega L(\omega))(\ G_{dc} + j\omega C)}$$

$$k(G_{dc}= 0) =\ -j\sqrt{(R(\omega)+j\omega L(\omega))(\ j\omega C)}\ .$$

The point is that the value of k *changes* when G_{dc} is turned on. The k ratio is then

$$\frac{k(G_{dc}> 0)}{k(G_{dc}= 0)}\ = \sqrt{\frac{G_{dc} + j\omega C}{j\omega C}}\ .$$

We can then rewrite the above theory alteration for turning on the dielectric conductivity in this manner

$$I(G_{dc}= 0) = 2\pi a\ \{N_0\}\ \frac{\omega}{k(G_{dc}=0)}$$

$$I(G_{dc}> 0) = 2\pi a\ \{(\xi_d/\varepsilon_d)N_0\}\ \frac{\omega}{k(G_{dc}>0)}\ .$$

The current ratio is then

$$\frac{I(G_{dc}> 0)}{I(G_{dc}= 0)}\ = (\xi_d/\varepsilon_d)\ \frac{k(G_{dc}= 0)}{k(G_{dc}> 0)}\ =\ (\xi_d/\varepsilon_d)\sqrt{\frac{j\omega C}{G_{dc} + j\omega C}}\ .$$

But from (1.5.1a) we know that

$$\xi_d \equiv \varepsilon_d - j\sigma_d/\omega \qquad \Rightarrow \qquad (\xi_d/\varepsilon_d)\ = 1 - j\ (\sigma_d/\varepsilon_d)(1/\omega)$$

Then using (4.4.10)

$$C/G_{dc} = \varepsilon_d/\sigma_d \qquad (4.4.10)$$

we find that

$$(\xi_d/\varepsilon_d) = 1 - j\,(G_{dc}/C)(1/\omega) = 1 + G_{dc}/(j\omega C) = \frac{G_{dc} + j\omega C}{j\omega C} \ .$$

Therefore our current ratio becomes,

$$\frac{I(G_{dc} > 0)}{I(G_{dc} = 0)} = (\xi_d/\varepsilon_d)\sqrt{\frac{j\omega C}{G_{dc} + j\omega C}} = \frac{G_{dc} + j\omega C}{j\omega C}\sqrt{\frac{j\omega C}{G_{dc} + j\omega C}} = \sqrt{\frac{G_{dc} + j\omega C}{j\omega C}}$$

$$= \sqrt{\xi_d/\varepsilon_d}$$

Here then is our rule: The effect of moving from $G_{dc} = 0$ to $G_{dc} > 0$ on the Appendix D fields is that one should take $I \rightarrow \sqrt{\xi_d/\varepsilon_d}\, I$. In other words, all the E fields become "larger" by the factor $\sqrt{\xi_d/\varepsilon_d}$. Since the B fields are obtained from $-j\omega\mathbf{B} = \text{curl}\mathbf{E}$ as in Section D.4, it follows that the B fields also become larger by this same factor $\sqrt{\xi_d/\varepsilon_d}$.

Another way to express this rule is this

$$I = 2\pi\omega\,(a/k)\,N_0 \qquad \rightarrow \qquad I' = 2\pi a\,(\omega/k')\,(\xi_d/\varepsilon_d)\,N_0 \qquad (D.2.31a)$$

or

$$I = (\omega/k)\,CV \qquad \rightarrow \qquad I' = (\omega/k')\,CV \qquad (D.2.31c)$$

where I and k are for $G_{dc} = 0$, and I' and k' are for $G_{dc} > 0$.

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Now let's start over on our Paradox!

1A. On the one hand, I have argued in Appendix D.9 that when we change from G = 0 to G > 0, the Appendix D solutions change simply by the addition of the factor (ξ_2/ε_2) to all the E and B fields and changing k to k'. For example, when G = 0 the E field solutions are,

Second summary of the E field solutions : (D.2.33)

$$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\ (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\ (ak)\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

$$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\ (ak)\ h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

where

$$I = 2\pi a\,(\omega/k)\,N_0 = 2\pi a\,(\omega/k)\,(1/2\pi a)\,q = (\omega/k)\,CV \quad . \qquad \text{(D.2.31c)}$$

Thus, for G = 0 the E fields are

$$E_z(r,m) = (1/4)\,\eta_m\ (\omega/k)\,CV\,R_{dc}\ (a\beta')\,f_m$$
$$E_r(r,m) = (j/4)\,\eta_m\ (\omega/k)\,CV\ R_{dc}\ (ak)\,g_m$$
$$E_\theta(r,m) = (1/4)\,\eta_m\ (\omega/k)\,CV\ R_{dc}\ (ak)\ h_m$$

and for G > 0 the fields are

$$E_z(r,m) = (\xi_2/\varepsilon_2)\,(1/4)\,\eta_m\ (\omega/k')\,CV\,R_{dc}\ (a\beta')\,f_m$$
$$E_r(r,m) = (\xi_2/\varepsilon_2)\,(j/4)\,\eta_m\ (\omega/k')\,CV\ R_{dc}\ (ak)\,g_m$$
$$E_\theta(r,m) = (\xi_2/\varepsilon_2)\,(1/4)\,\eta_m\ (\omega/k')\,CV\ R_{dc}\ (ak)\ h_m$$

where k' is given by the usual formula but with G > 0, whereas k is the same formula with G = 0.

The entire change can be encapsulated by saying

$$I = (\omega/k)\,CV \qquad \rightarrow \qquad I = (\omega/k')\,CV(\xi_2/\varepsilon_2)$$

As shown above, the current ratio for making this change is:

$$I_{G>0} / I_{G=0} = \sqrt{\xi_d/\varepsilon_d} = \sqrt{\frac{G_{dc} + j\omega C}{j\omega C}} \qquad (*)$$

2A. On the other hand, I know that

$$1/Z_0 = \sqrt{\frac{G + j\omega C}{R + j\omega L}}$$

where for low ω we have $R = R_{dc2}$. I also know that

$$I = V/Z_0$$

Therefore

$$I = V\sqrt{\frac{G + j\omega C}{R + j\omega L}} \qquad G > 0$$

For G = 0 this reads

$$I = V\sqrt{\frac{j\omega C}{R + j\omega L}} \qquad G = 0$$

The ratio by which I increases is this:

$$I_{G>0} / I_{G=0} = \sqrt{\frac{G + j\omega C}{R + j\omega L}} \Big/ \sqrt{\frac{j\omega C}{R + j\omega L}} = \sqrt{\frac{G+j\omega C}{j\omega C}} = \sqrt{\xi_d/\varepsilon_d} \qquad (**)$$

3A. The Paradox Resolved: Comparing these two results (*) and (**), we see that they are exactly the same and thus there is no paradox!

Comments: this business with the symbol I is quite confusing. It appears in my second summary of the E fields in (D.2.33) for example. The problem is that k also appears in these formulas, and you have to remember to take k → k'. so the fields in general *do not* do a simple scaling! For example,.

Second summary of the E field solutions : $R_{dc} = \frac{1}{\sigma\pi a^2}$ $\beta'^2 = \beta^2 - k^2$ (D.2.33)

$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta')\, f_m$ $\quad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right]$ $\quad x = \beta' r$

$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (ak)\, g_m$ $\quad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$ $\quad x_a = \beta' a$

$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (ak)\, h_m$ $\quad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$

There is a k hidden inside the f_m factors for example, and k → k'. There are also some explicit k values sitting there, and they too must do k→ k'. NOW, if we think of

$$I = (\omega/k)\, CV$$

then there is no N_0 appearing, so our only rule is to take k→k' I guess. That in itself creates the $\sqrt{\xi_d/\varepsilon_d}$ factor in going I → I', since ω and V and C don't change.

But the problem is that I always just write k as k, and its value then goes in through the formula when I eventually apply the k(ω) formula. So this is all very confusing and needs to be cleaned up !!!

It is true that the actual physical current in the wire increases by the factor $\sqrt{\xi_d/\varepsilon_d}$.

Next day 10.6.14. I am now thinking about how to rewrite the text after (D.9.24). Several possibilities:

1. If I don't distinguish the change in k as k', then (D.9.24) is extremely unclear. You get the impression that the current changes by the full factor (ξ_d/ε_d). I could say

$$I = 2\pi a N_0(\omega/k) \quad \rightarrow \quad I' = 2\pi a N_0(\omega/k')\,(\xi_d/\varepsilon_d)\,. \tag{D.9.24}$$

where k applies to the G = 0 situation and k' to the G>0 situation. But then I have to segue to the notion that k is not a fixed constant, but k = k(ω) = our formula, and then k' ≠ k. The prime might be too lightweight.