Rewrite of Section D.11 PhL 6.29.14

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D.11 Low frequency limit of the round wire E fields

(a) A High Level Review of Appendix D and its Accuracy

As presented above, the general approach of Appendix D was to solve for the E and B fields inside a round wire assuming the ansatz traveling wave form

$$\mathbf{E}(r,\theta,z,t) = e^{j(\omega t - kz)}\,\mathbf{E}(r,\theta)\ , \qquad (D.1.1)$$

where k is an arbitrary complex parameter. For *any k*, we found the following **E** field solution, where we now show k dependence more explicitly:

Second summary of the E field solutions : $R_{dc} = \frac{1}{\sigma\pi a^2}$ $\beta'(k)^2 = \beta^2 - k^2$ *(D.2.33)*

$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\ [a\beta'(k)]\, f_m(k)$ $\quad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right]$ $\quad x = \beta'(k)r$

$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\ (ak)\, g_m(k)$ $\quad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$ $\quad x_a = \beta'(k)a$

$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\ (ak)\ h_m(k)$ $\quad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$

These fields exactly solve Maxwell's equations and the two boundary conditions (D.2.26) and (D.2.27), and from these E fields we computed the corresponding B fields. The coefficients η_m are the moments of the surface charge distribution n(θ) on the round wire surface. In principle, any linear combination of these solutions for different k values (including a continuous superposition) is also a possible solution.

However, when this round wire is part of a transmission line, we must also take into consideration the field solution outside the round wire -- the solution within the transmission line dielectric region. This is the so-called exterior solution, whereas our round wire analysis provided an interior solution. The idea is that the exterior solution provides the correct value of parameter k to use for the interior solution. The solutions must have the same k value due to the boundary between interior and exterior.

Whereas Appendix D found the interior solution for the **E** field using the Helmholtz equation, Chapters 3 and 4 obtained the exterior solution in terms of the potentials φ and A_z using the King gauge condition. This analysis was not valid at low frequencies for a variety of reasons noted in those chapters,

perhaps the most dramatic of which is shown in Fig 3.6.(b). This drawing illustrates how the round wires of a twin-lead transmission line are clearly not surfaces of constant A_z potential at very low frequency, whereas the theory assumes that they are. The main results of Chapter 4 were the first and second order "transmission line equations" (4.11.14b) and (4.11.15) involving i(z) and V(z). The second order equations are (damped) wave equations which directly imply an e^{-jkz} dependence on z. Through the boundary between the interior and interior solutions, this implies a similar e^{-jkz} form for the interior solutions, which form is the ansatz of Appendix D. However, at low frequencies these wave equations are no longer valid, there are "correction terms", and thus the e^{-jkz} ansatz (D.1.1) of Appendix D is no longer valid. Therefore, we cannot expect low frequency predictions of Appendix D concerning interior fields to be accurate.

Meanwhile, on a separate track altogether, Appendix K describes the so-called "network model" of the exterior solution [at least i(z) and V(z)] for a transmission line, using lumped R,G,L,C components. In this model, the same transmission line equations obtained in Chapter 4 are found to be true, justifying the network model. However, in the network model, these transmission line equations are valid all the way down to DC (ω=0) whereas we have just shown that the "physics model" does not support this conclusion. Nevertheless, we can use the network model's low frequency range as an approximation to the true exterior solution at low frequency. In other words, we can pretend that the transmission line equations are valid all the way down to DC. In so doing, we should not be surprised to find results which are inaccurate. Note that the network model says nothing about interior field solutions.

Above low frequencies both the physics and network models provide the same value of k to be used in the round wire interior solution. That value is $k = -j\sqrt{zy} = -j\sqrt{(R+j\omega L)(G+j\omega C)}$. Since k is a function of ω, the transmission line has "dispersion" and a group velocity $v_g = \partial\omega/\partial k$ different from the phase velocity $v_\varphi = \omega/k$. Appendix Q obtains expressions for k(ω) appropriate for both high and low frequencies as limits of this rather complicated function. We then use these limits, knowing that they can give inaccurate results, in our low frequency analysis below.

(b) Low frequency values for β'

For low ω and G > 0, we use these expressions for β and k,

$\beta^2 = -j\omega\mu\sigma$ *(1.5.1a)* for good conductor

$k \approx (\omega/2)\dfrac{RC+GL}{\sqrt{RG}} - j\sqrt{RG}$. *(Q.6)*

so that at low ω

$\beta^2 \approx 0$

$k \approx -j\sqrt{RG}$ => $k^2 = -RG$

$\beta'^2 = \beta^2 - k^2 = -k^2 \approx RG$. (D.11.1)

For any reasonable transmission line RG will very small so the Bessel argument $x_a = \beta' a << 1$.

Example: Belden 8281 coaxial cable has radius a = 394μ, K = 3.7 and R = 38.5 ohms/km ≈ .04 ohm/m. At a worst case 100 GHz it has a significant $\sigma_d = \sigma_{eff} =$ 2.6 x 10^{-3} mho/m from (3.3.6). From (4.11.34) $G = 4\pi\sigma_d/K = 4\pi$ 2.6 x 10^{-3}/ K = .009 mho/m. Then RG = .04*.009 = .00036 and $\beta' = \sqrt{RG} \approx .02\ m^{-1}$. Finally, the maximum Bessel function argument in (D.2.33) is $x_a = \beta' a$ = .02 * 394e-6 ≈ 10^{-5}.

Since β'a is very small, we shall need to evaluate f_m, g_m and h_m for small β'.

For low ω and G= 0, we use instead these expressions

$\beta^2 = -j\omega\mu\sigma$ *(1.5.1a)* for good conductor

$k \approx \sqrt{RC/2}\ \omega^{1/2}(1-j) = \sqrt{RC}\ \omega^{1/2} e^{-j\pi/4}$ *(Q.7)* (highly damped)

$k^2 = RC\omega(-j) = -j\omega RC$

$\beta'^2 = \beta^2 - k^2 \approx -j\omega\mu\sigma + j\omega RC = -j\omega(\mu\sigma - RC)$ (D.11.2)

Again β' is very small at low frequency.

(c) Low frequency evaluation of f_m, g_m and h_m

In the following we consider only m ≥ 0 since we know from (D.10.2) that $f_{-m} = f_m$, $g_{-m} = g_m$, $h_{-m} = h_m$.

The small x limit for $J_m(x)$ is given by NIST 10.7.3,

$J_n(x) = (x/2)^n / n!$. for n = 0,1,2,..... (D.11.3)

Since J_{m-1} appears in our coefficient expressions and since m = 0 is encountered, we have to deal with m = 0 as a special case since the above limit is not valid for n = -1. To this end we use NIST 10.2.2 which is valid for integer n,

$J_{-n}(x) = (-1)^n J_n(x) \approx (-1)^n (x/2)^n / n!$ (D.11.4)

so that $J_{-1}(x) = -J_1(x) \approx -(x/2)$. Our small-x forms of interest are then

$J_n(x) = (x/2)^n / n!$ for n = 0,1,2,.....
$J_{-1}(x) = -(x/2)$ for n = -1 . (D.11.5)

We now examine the small x limits of f_m, g_m, and h_m .

First f_m for m > 0, and then for m = 0:

$$f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] = \left[\frac{(x/2)^m/m!}{(x_a/2)^{m+1}/(m+1)!} - \frac{(x/2)^m/m!}{(x_a/2)^{m-1}/(m-1)!}\right]$$

$$= [\,(m+1)\,(x/x_a)^m\,(2/x_a) \;-\; (1/m)\,(x/x_a)^m (x_a/2)\,]$$

$$= (x/x_a)^m\,[\,(m+1)\,(2/x_a) \;-\; (1/m)\,(x_a/2)\,]$$

$$\approx (x/x_a)^m\,(m+1)\,(2/x_a) \qquad \text{// as } x_a \to 0$$

$$f_0 = \left[\frac{J_0(x)}{J_1(x_a)} - \frac{J_0(x)}{J_{-1}(x_a)}\right] = \left[\frac{J_0(x)}{J_1(x_a)} + \frac{J_0(x)}{J_1(x_a)}\right] = 2\,\frac{J_0(x)}{J_1(x_a)} = 2\,\frac{1}{(x_a/2)} = 4/x_a$$

First g_m for $m > 0$, and then for $m = 0$:

$$g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] = \left[\frac{(x/2)^{m+1}/(m+1)!}{(x_a/2)^{m+1}/(m+1)!} + \frac{(x/2)^{m-1}/(m-1)!}{(x_a/2)^{m-1}/(m-1)!}\right]$$

$$= (x/x_a)^{m+1} + (x/x_a)^{m-1}$$

$$g_0 = \left[\frac{J_1(x)}{J_1(x_a)} + \frac{J_{-1}(x)}{J_{-1}(x_a)}\right] = \left[\frac{J_1(x)}{J_1(x_a)} + \frac{J_1(x)}{J_1(x_a)}\right] = 2\,\frac{J_1(x)}{J_1(x_a)} = 2\,(x/x_a)$$

Results for h_m are then obvious since there is only a sign change between the terms in g_m,

$$h_m = (x/x_a)^{m+1} - (x/x_a)^{m-1} \qquad h_0 = 0$$

The results are then,

$$\begin{aligned} f_m &= (r/a)^m\,(m+1)\,(2/\beta' a) & f_0 &= 4/(a\beta') \\ g_m &= (r/a)^{m+1} + (r/a)^{m-1} & g_0 &= 2\,(r/a) \\ h_m &= (r/a)^{m+1} - (r/a)^{m-1} & h_0 &= 0 \end{aligned}$$

for $m \geq 0$. Allowing for all integer values of m, using the symmetries (D.10.2) we can write,

$$\begin{aligned} f_m &= (r/a)^{|m|}\,(|m|+1)\,(2/\beta' a) & f_0 &= 4/(a\beta') \\ g_m &= (r/a)^{|m|+1} + (r/a)^{|m|-1} & g_0 &= 2\,(r/a) \\ h_m &= (r/a)^{|m|+1} - (r/a)^{|m|-1} & h_0 &= 0 \quad . \end{aligned} \qquad \text{(D.11.6)}$$

The fields in (D.2.33) quoted above then become,

$$\begin{aligned} E_z(r,m) &= (1/4)\,\eta_m\, I\, R_{dc}\; [a\beta']\, f_m & f_m &= (r/a)^{|m|}\,(|m|+1)\,(2/\beta' a) & f_0 &= 4/(a\beta') \\ E_r(r,m) &= (j/4)\,\eta_m\, I\, R_{dc}\; (ak)\, g_m & g_m &= (r/a)^{|m|+1} + (r/a)^{|m|-1} & g_0 &= 2\,(r/a) \\ E_\theta(r,m) &= (1/4)\,\eta_m\, I\, R_{dc}\; (ak)\; h_m & h_m &= (r/a)^{|m|+1} - (r/a)^{|m|-1} & h_0 &= 0 \end{aligned}$$

or

$$E_z(r,m) = (1/2)\,\eta_m\, I\, R_{dc}\, (r/a)^{|m|}\, (|m|+1)$$
$$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (ak)\, [(r/a)^{|m|+1} + (r/a)^{|m|-1}]$$
$$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (ak)\, [(r/a)^{|m|+1} - (r/a)^{|m|-1}]$$

$$E_z(r,0) = I\, R_{dc}$$
$$E_r(r,0) = (j/2)\, I\, R_{dc}\, (ak)\, (r/a)$$
$$E_\theta(r,0) = 0 \qquad \text{// low } \omega \text{ E fields} \qquad (D.11.7)$$

If $n(\theta)$ is real and even in θ, we know from (D.10.4a) that,

$$E_z(r,\theta) = (1/4)\, I\, R_{dc}\, (a\beta')\, [\, f_0 + 2\, \Sigma_{m=1}^{\infty}\, f_m\, \eta_m\, \cos(m\theta)\,]$$
$$E_r(r,\theta) = (j/4)\, I\, R_{dc}\, (ak)\, [\, g_0 + 2\, \Sigma_{m=1}^{\infty}\, g_m\, \eta_m\, \cos(m\theta)\,]$$
$$E_\theta(r,\theta) = (1/4)\, I\, R_{dc}\, (ak)\, [\, h_0 + 2\, \Sigma_{m=1}^{\infty}\, h_m\, \eta_m\, \cos(m\theta)\,] \ . \qquad \textit{(D.10.4a)}$$

Inserting the expressions (D.11.6) then gives

$$E_z(r,\theta) = I\, R_{dc}\, \{\, 1 + \Sigma_{m=1}^{\infty}\, (r/a)^m\, (m+1)\, \eta_m\, \cos(m\theta)\,\}$$
$$E_r(r,\theta) = (j/2)\, I\, R_{dc}\, (ak)\, \{(r/a) + \Sigma_{m=1}^{\infty}\, [(r/a)^{m+1} + (r/a)^{m-1}]\, \eta_m\, \cos(m\theta)\,\}$$
$$E_\theta(r,\theta) = (1/2)\, I\, R_{dc}\, (ak)\, \{\, 0 + \Sigma_{m=1}^{\infty}\, [(r/a)^{m+1} - (r/a)^{m-1}]\, \eta_m\, \cos(m\theta)\,\} \ . \qquad (D.11.8)$$

Notice that

$$\frac{E_z(r,\theta)}{E_z(r,0)} = 1 + \Sigma_{m=1}^{\infty}\, (r/a)^m\, (m+1)\, \eta_m\, \cos(m\theta) \ . \qquad (D.11.9)$$

An anomaly. This last result is supposedly valid for very low ω, and we see that $E_z(r,\theta)$ and the above ratio are independent of ω since k does not appear anywhere. We would expect that in the limit $\omega \to 0$ the above ratio should be exactly 1, at least for a non-conducting dielectric. This is so because we expect there to be no eddy currents at $\omega = 0$ at these are the cause of J_z non-uniformity as discussed in Appendix P. We attribute this anomalous result to the inaccuracy of the model at low ω, as outlined in section (a) above. It happens that in the limit $\omega \to 0$ we also have $I \to 0$ when $G = 0$ (since then $Z_0 = \infty$), but that is no justification for the anomalous result.

(d) Low frequency E fields

We ignore the above anomaly and proceed with our task of finding the low frequency E fields, knowing that it is all just an approximation. Recall from (4.11.16) that

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \qquad 1/Z_0 = \sqrt{\frac{G + j\omega C}{R + j\omega L}} \ . \qquad \textit{(4.11.16)}$$

For low ω we find from this expression for $1/Z_0$ and from (D.11.1,2) that,

$$I = V/Z_0 \approx V\sqrt{G/R} \qquad k \approx -j\sqrt{RG} \qquad G > 0 \qquad \text{(D.11.10)}$$

$$I = V/Z_0 \approx V\sqrt{j\omega C/R} \qquad k \approx \sqrt{RC}\,\omega^{1/2}\,e^{-j\pi/4} \qquad G = 0 \qquad \text{(D.11.11)}$$

Inserting (D.11.10) into (D.11.7) gives, for G > 0,

Small ω limit of the E field solutions, G>0 : $R_{dc} = \dfrac{1}{\sigma\pi a^2}$ (D.11.12)

$$E_z(r,m) = (1/2)\,\eta_m\,V\,R_{dc}\sqrt{G/R}\;(r/a)^{|m|}\,(|m|+1) \qquad Z_0 = \sqrt{R/G}$$
$$E_r(r,m) = a\,(1/4)\,\eta_m\,V\,R_{dc}\,G\,[(r/a)^{|m|+1} + (r/a)^{|m|-1}]$$
$$E_\theta(r,m) = -ja\,(1/4)\,\eta_m\,V\,R_{dc}\,G\,[(r/a)^{|m|+1} - (r/a)^{|m|-1}] \qquad k = -j\sqrt{RG}$$

$$E_z(r,0) = V\,R_{dc}\sqrt{G/R}$$
$$E_r(r,0) = a\,(1/2)\,V\,R_{dc}\,G\,(r/a)$$
$$E_\theta(r,0) = 0$$

The fields are all finite and there are non-zero expressions for E_r and E_θ which account for the expected non-uniform flow of current into the dielectric through the conductor boundaries. We expect that $E_r(r,\theta)$ just outside the conductor boundary is a strong function of θ for closely spaced conductors (the capacitor problem), and thus so is $J_r(r,\theta)$. But $J_r(a,\theta)$ is continuous through the boundary at ω = 0, so we expect to see a strong dependence of $J_r(a,\theta)$ on θ inside the round wire, as indicated by $E_r(r,m)$ in (D.11.12).

Reader Exercise: Does (D.11.12) give the correct solution to the implied magnetostatics problem, or are there anomalies like the one noted above? Notice that Z_0 is certainly correct based on the reader exercise given in Section K (c). The "wave" decays in z according to $e^{-jkz} = \exp(-\sqrt{RG}\,z)$ which also seems reasonable.

Next, inserting (D.11.11) into (D.11.7) gives, for G = 0, this limiting form :

Small ω limit of the E field solutions, G=0 : $R_{dc} = \dfrac{1}{\sigma\pi a^2}$ (D.11.13)

$$E_z(r,m) = (1/2)\,\eta_m\,V\,\omega^{1/2}\sqrt{jC/R}\;R_{dc}\,(r/a)^{|m|}\,(|m|+1)$$
$$E_r(r,m) = a\,(j/4)\,\eta_m\,V\,\omega\,C\;R_{dc}\;[(r/a)^{|m|+1} + (r/a)^{|m|-1}]$$
$$E_\theta(r,m) = a\,(1/4)\,\eta_m\,V\,\omega\,C\;R_{dc}\;[(r/a)^{|m|+1} - (r/a)^{|m|-1}]$$

$$E_z(r,0) = V\,\omega^{1/2}\sqrt{jC/R}\;R_{dc}$$
$$E_r(r,0) = a\,(j/2)\,V\,\omega\,C\;R_{dc}\,(r/a)$$
$$E_\theta(r,0) = 0$$

As $\omega \to 0$, all fields vanish, corresponding to the fact that $Z_0 = \sqrt{R/j\omega C} \to \infty$ so $I = V/Z_0 \to 0$. In this situation the network model is just an infinite ladder of series resistors with no conductance cross pieces.

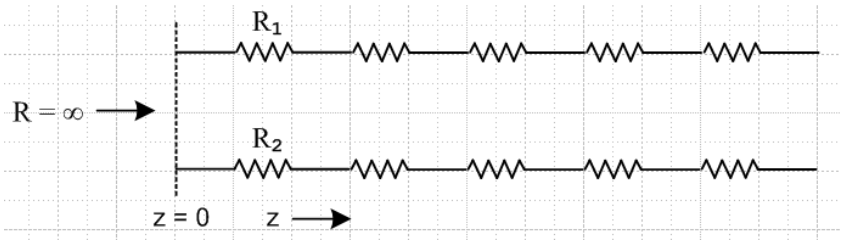


Fig D.8