Appendix D Section D.8 PhL 12.13.13

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D.8 About the boundary condition $E_\varphi(r=a,m) = 0$

In earlier sections of this Appendix we examined the electric field inside a round wire (radius a) which was regarded as a conductor in a straight transmission line. The electric field was assumed to have the form of a longitudinal wave travelling down the conductor,

$$\mathbf{E}(r,\varphi,z,t) = e^{j(\omega t-\beta_d z)}\,\mathbf{E}(r,\varphi)\,, \qquad (D.1.2)$$

where β_d is the wavenumber parameter of the surrounding dielectric medium. We expanded the function $\mathbf{E}(r,\varphi)$ onto azimuthal partial waves $e^{jm\varphi}$ and solved the Helmholtz wave equation inside the wire with solutions as shown in (D.2.21),

$$E_z(r,m) = -j\,(\beta'/\beta_d)\,\frac{K_m}{2}\,J_m(x) \qquad x = \beta' r \qquad (D.1.27)$$

$$E_r(r,m) = a_m\,x^{-1}\,J_m(x) + \frac{K_m}{2}\,J_{m+1}(x)\,. \qquad (D.2.11)$$

$$jE_\varphi(r,m) = -a_m\,x^{-1}\,J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x)\,. \qquad (D.2.15)$$

where $\beta'^2 = \beta^2-\beta_d^2$ with β being the (complex) wavenumber parameter of the conductor, and where a_m and K_m are undetermined constants.

At this point we applied the two boundary conditions,

$$E_r(r=a,m) = (j\omega/\sigma)\,N_m \qquad (D.2.26)$$

$$E_\varphi(r=a,m) = 0\,. \qquad (D.2.27)$$

where N_m is the m^{th} partial wave moment of the surface charge $n(\varphi)$ distribution, where

$$n(\varphi,z,t) = e^{j(\omega t-\beta_d z)}\,n(\varphi)\,. \qquad (D.1.4)$$

These conditions determined the constants a_m and K_m giving the resulting $\mathbf{E}$ field inside the wire,

$$E_z(r,m) = (1/4)\,\eta_m\,I\,R_{dc}\,\left[\frac{x_a J_m(x)}{J_{m+1}(x_a)} - \frac{x_a J_m(x)}{J_{m-1}(x_a)}\right] \qquad a = \text{radius} \quad \eta_m \equiv \frac{N_m}{N_0} \qquad (D.2.33)$$

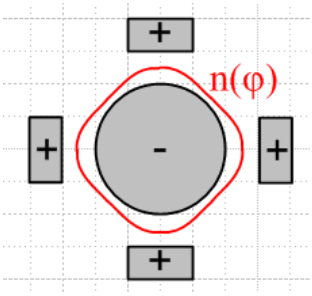
$$E_r(r,m) = (j/4)\,\eta_m\,I\,R_{dc}\,(a\beta_d)\,\left[\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m+1}(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E_\varphi(r,m) = (1/4)\,\eta_m\,I\,R_{dc}\,(a\beta_d)\,\left[-\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m+1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

where $R_{dc} = 1/(\pi a^2 \sigma)$ is the DC resistance of the wire per unit length, and I is the amplitude of the current in the wire. Everything is an implicit function of frequency ω. It can be seen that in the transmission line limit $(a\beta_d) << 1$ the fields E_r and E_φ are much smaller than E_z.

An implication of the solution is that the E fields inside the wire for each partial wave are described by a single parameter N_m which is the surface charge moment noted above. If the other transmission line conductor(s) were to change their position relative to the round wire and/or to vary their cross sectional shape, the only effect this would have would be to adjust the set of parameters N_m, and the solutions would still be given by (D.2.23) quoted above. Although the set $\{N_m\}$ is infinite, it seems likely that for reasonable shapes of the other conductor(s), the lowest few N_m partial waves would provide a good approximation to the E fields inside the wire. Since Ohm's Law is assumed to apply inside the wire, one then knows in detail the current densities J_z, J_r and J_z. The magnetic field **B** inside the wire is then also known and was calculated above. The lowest moment is always $N_0 = (\beta_d/2\pi\omega a)\, I$ from (D.2.31).

As a passing example, the following five-conductor transmission line might be expected to have a strong m = 2 quadrupole surface charge moment N_2,



A critical ingredient of our solution is the assumption that $E_\varphi(r=a,m) = 0$ and that is the subject now addressed. We present two somewhat different arguments as to why $E_\varphi(r=a,m) = 0$. It should be noted that King in his *Transmission-Line Theory* book always assumes that any straight transmission line conductor cross section has an equipotential surface (a ring, see for example middle p 14, top 15, 25 bottom).

(a) The Quasi-Static Argument

In electrostatics, we are used to metal surfaces being equipotentials. For example, if we put a point charge q near a metal sphere, it induces a surface charge on that sphere. The electric field lines land on the sphere exactly perpendicular to the surface. One argues that if there were even some tiny E field component tangential to the surface, the surface charges would adjust their position to cancel out that tangential field. Since the situation is static, any adjustment has already been made. Since $E_{tan} = 0$, the sphere's surface is an equipotential surface.

If we were to then slowly move the charge q around (perhaps it rotates in a circle around the sphere), the surface charge instantly adjusts at each new position of q, and those E field lines remain perpendicular to the surface, and $E_{tan} = 0$. While the charges are adjusting position, there is admittedly some very tiny surface current driven by some tiny E_{tan} , But if we move the charge slowly, we are "quasi-static" and the approximation $E_{tan} \approx 0$ is very good. One might compare the time constant of the moving sphere (T, the period of q's revolution around the sphere) to the time constant of the surface charge adjustment. For copper the time constant is roughly the mean electron collision time which is on the order of 10^{-14} sec.

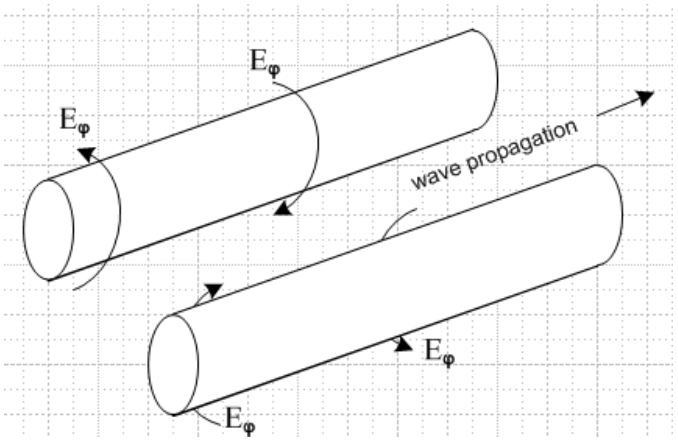
The upshot here is that for frequencies << 10^{14} Hz, the quasi-static situation prevails and then $E_{tan} \approx 0$ is a very good approximation.

This then is our first argument for why we claim the boundary condition $E_\varphi = 0$ on the surface of the round wire in a transmission line operating at a typical frequency.

We note from our solution $E_\varphi(r,m)$ that if we assume $E_\varphi(a,m) = 0$ on the round wire surface, we will still have $E_\varphi(r,m) \neq 0$ inside the wire. This fact is consistent with our argument above since there are no free charges available to adjust themselves inside the wire.

However: if $E_\varphi = 0$ by this quasi-static argument, then we should expect that $E_z = 0$ by the same argument, since E_z is also a tangential field at the round wire surface, and since E_z operates at the same frequency ω as E_φ. But we know that $E_z \neq 0$ because $J_z \neq 0$ just below the wire surface -- there is current flowing there -- and E_z is continuous through the surface by (1.1.50). So the E field lines are not quite perpendicular to the round wire surface in the z direction. This is not too surprising since we expect everything to vary in the z direction as $e^{j(\omega t - \beta_d z)}$ so we would expect the surface not to be an equipotential in this direction.

But what happened to that quasi-static argument we applied to E_φ ? What happened is that there is external field activity associated with the wave going down the line which forces $E_z \neq 0$. One might say the EM wave travelling down the line induces a J_z in the round wire, with its associated $E_z \neq 0$. But then perhaps this same thing could somehow happen with E_φ and then our quasi-static argument that $E_\varphi = 0$ collapses. We think this could happen in fact, but only if the transmission line is driven by an apparatus which creates a "torsion wave" in the line. For example, the apparatus could drive counter-rotating azimuthal currents onto the round wire surfaces of a twin-lead transmission line as suggested by this picture (which is not meant to imply that other field components vanish),



It seems from our work above that such a wave would satisfy Maxwell's equations and be a viable mode of the transmission line. In this case, $E_\varphi \neq 0$ because the EM wave going down the line forces $E_\varphi \neq 0$, just as the normal wave forces $E_z \neq 0$.

We have not investigated whether this type of torsion wave is really viable. Whether or not it is, we assume in our transmission line discussion that this mode is not activated and that therefore the quasi-static argument for $E_\varphi = 0$ is valid at the round wire surface.

(a) An Ansatz Argument

We are always assuming our transmission line operates in the transmission line limit $\beta_d d << 1$ where d is any transverse dimension of the line. We anticipate that in this limit, E_r and E_φ are much smaller than E_z. We assume this is true, and see if this assumption is born out in a final solution of Maxwell's equations.

Given that E_φ is very small, we can make an approximation (an ansatz) that this field E_φ is exactly zero on the surface of the round wire. This may not be exactly true, but we assume it for our purposes and see where it leads. This is the nature of an "ansatz".

When we make this assumption, the cross section of the transmission line may be regarded as a two dimensional potential theory problem, basically a capacitor problem where one conductor has potential V and the other -V, say. In such a potential problem, one always assumes that the electrostatic potential Φ is a constant on the surface of each conductor, and that is precisely what our ansatz says: $E_\varphi = -(\nabla\Phi)_\varphi = 0$, Φ = constant in the φ direction. Now when we solve the capacitor problem for potential Φ, that gives $\mathbf{E} = -\nabla\Phi$ in the dielectric between the conductors, and from that we may deduce $\mathbf{E} \bullet \hat{\mathbf{n}}$ at the surface of one of the conductors. For the round wire with a cylindrical coordinate system whose axis is aligned with the wire center, that field is E_r . Next, from this surface value of E_r (which will be proportional to V) we may compute the surface charge density $n(\varphi)$ on the round wire using (D.1.22) which says $E_r(r=a,\varphi) = (j\omega/\sigma)\, n(\varphi)$. For a "fat" twin lead transmission line for example we expect this to have a bulge in $n(\varphi)$ on the side of the wire facing the other wire (m = 1, dipole), since that is what happens in such a capacitor. In any event, given $n(\varphi)$ we may compute the moments N_m of the surface charge using (D.1.6) and this then provides one "boundary condition" on our coefficients a_m and K_m which appear in all the field expressions we found above,

$$E_r(r=a,m) = (j\omega/\sigma)\, N_m \,. \qquad \text{(D.2.18)}$$

But recall that, in order to carry out this entire process just described, we had to start with the assumption that $E_\varphi = 0$ on the conductor cross section surface, so that we could have a capacitor problem in the first place. According to the right equation of (D.1.3), if $E_\varphi(r=a,\varphi) = 0$, then $E_\varphi(r=a,m) = 0$, so that in fact we must have $E_r(a,m)$ being zero in all partial waves m. Thus our assumed ansatz condition is

$$E_\varphi(r=a,m) = 0 \qquad \text{(D.2.19)}$$

which is then a second boundary condition on a_m and K_m. Although (D.2.19) might not be exactly true, we know it is very close to being true. More importantly, we know that the above two conditions on a_m and K_m are consistent with each other, even though both boundary conditions might be slightly wrong. We then expect them to give good values for constants a_m and K_m in the transmission line limit where the ansatz (D.2.19) is very reasonable.

There are two footnotes that one might add to the above discussion.

First, we note that the second boundary condition does not force $E_\varphi(r,m) = 0$ for $r < a$ inside the wire. In fact, there will be some small azimuthal "swirling" current inside the wire even if $E_\varphi(r=a,m) = 0$, and this is just a result of Maxwell's equations and their solutions above. One might make the alternative argument that the round wire surface is an equipotential since that is the way a line is driven at the source. For example, the center conductor of a coaxial cable plugs into a tiny driving cylinder (jack) in a BNC connector and this drives only the wire surface, and it does so in an azimuthally symmetric way so that one expects to have the wire surface be an equipotential at the driving point, and this equipotential surface

then moves down the line as the wave progresses. But we know that things work just fine if the center conductor is driven instead by an abutting gold block, say, and such a driving mechanism would force the potential to be constant across the entire round face of the wire, which contradicts the fact that there must be those swirling currents in the wire interior. In this case, the swirling currents would in fact be zero at the driving gold block, but they would soon develop after a short transition distance in z. So it seems that the best way to understand (D.2.19) is in the self-consistent approximation sense outlined above.

Second, we have the complication that we don't really have a purely electrostatic situation, and the potential is in fact related to **E** by equation (1.3.1) which says $\mathbf{E} = -\nabla\Phi - \partial_t\mathbf{A}$. The rescue here comes by claiming that roughly $\mathbf{A} \approx A\hat{\mathbf{z}}$ so that the transverse components A_r and A_φ are very small. In this case, we then do get $\mathbf{E} \approx -\nabla\Phi$ so that $E_\varphi = 0$ is associated with constant Φ on the wire surface. The argument for $\mathbf{A} \approx A\hat{\mathbf{z}}$ is that **A** is driven by **J**, and **J** is mostly in the $\hat{\mathbf{z}}$ direction, which in turn is related to our starting ansatz.

Exercise for the Reader

(1) Show that a classical electron inside or on the surface of a conductor of a transmission line traverses a tiny elliptical path of perhaps 1 nm scale and thus never really goes anywhere. That path is traversed once per period $T = 2\pi/\omega$. Mathematically, show that this amounts to proving that the three equations

$x = A\cos(\omega t - a)$
$y = B\cos(\omega t - b)$
$z = C\cos(\omega t - c)$

are parametric equations for an ellipse with some orientation in 3D space. This goes-nowhere aspect of the electron is similar to what happens with a droplet of water in an ocean wave. (Hint: first show that the first two equations describe an ellipse in the xy plane and that thee semi-major axes in general are *not* A and B .)

(2) When a TEM wave travels down a transmission line with a round conductor, the electric field "raises" a surface charge density on that conductor as it passes by. Exactly where does this surface charge come from? Is the charge density (though not individual charges) just sliding down the line in the z direction at the dielectric light velocity, and that is where it comes from -- the surface charge moves in the z direction, and there is then a z-directed surface current? Or does this charge get pumped off the other side of the conductor through the interior by the radial field E_r ? Or does the charge get driven around the cross section surface of the conductor by an E_φ field which we have proposed vanishes?

(3) Use Maple or other software to make a cross-sectional 2D "field plot" of current flow in a round wire for a given partial wave m. A starting point (for z = 0) might be

$$\mathbf{E}_{trans}(r,\varphi,t) = \cos(-\omega t + m\varphi + \arg[E_\varphi(r)])\,|E_\varphi|\;\hat{\boldsymbol{\varphi}} + \cos(-\omega t + m\varphi + \arg[E_r(r)])\,|E_r|\;\hat{\mathbf{r}}$$

where $\hat{\boldsymbol{\varphi}} = -\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}}$ and $\hat{\mathbf{r}} = \cos\varphi\,\hat{\mathbf{x}} + \sin\varphi\,\hat{\mathbf{y}}$. Make a series of plots at sequential t values to obtain a weather pattern for the E field components (and thus the currents), and see if this helps answer question (2) above. Try making a 3D field plot adding in the $\hat{\mathbf{z}}$ field component.