B field presentation PhL 9.20.14

Summary of this Doc: Here I set out to simplify the B field presentation in Appendix D. To do this, two new items have to be computed, since they appear in the curl E equations:

$$\partial_r f_m = \beta' \partial_x f_m(x)$$

$$\partial_r(rE_\theta) = (\beta' \partial_x)\,[\,(x/\beta')\, h_m(x)\,] = \partial_x\,(x\, h_m(x))$$

I show that

$$\partial_x\,(x\, h_m(x)) = x\, e_m - m\, g_m \qquad \text{where new} \qquad e_m \equiv \left[\frac{J_m(x)}{J_{m+1}(x_a)} + \frac{J_m(x)}{J_{m-1}(x_a)}\right]$$

$$\partial_x f_m = f_m\left[\,(m/x) - \frac{J_{m+1}(x)}{J_m(x)}\right]$$

I then end up with

$$B_r(r,m) = -(1/4)\,(a/\omega)\,\eta_m\, I\, R_{dc}\,\{\; r^{-1} m\;(\beta')\, f_m + k^2\; h_m\;\}$$

$$B_\theta(r,m) = (j/4)(a/\omega)\,\eta_m\, I\, R_{dc}\,\{k^2\, g_m - \beta'^2 f_m\,[\,(m/x) - J_{m+1}(x)/J_m(x)]\,\}$$

$$B_z(r,m) = (j/4)(a/\omega)\,\eta_m\, I\, R_{dc}\;\{k\,\beta'\, e_m\,\}$$

and this is how the results appear in new (D.4.13.), but in a different order.

I then start the task of using Maple to verify the above solutions, but that task is carried out in the main text and not below. I used an intermediate file "new section D_4 D_5 and D_6 on B fields.doc" along with the Maple file " B field verify 9_20.mws ".

I am not happy with the way I present the B fields in Appendix D. The E fields are so compact,

Second summary of the E field solutions : $R_{dc} = \dfrac{1}{\sigma\pi a^2}$ $\beta'^2 = \beta^2 - k^2$ (D.2.33)

$$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\;(a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\;(ak)\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

$$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\;(ak)\; h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

Can I come up with something compact like this for the B fields? After all, it is just this:

$-j\omega\,\mathbf{B} = \text{curl}\,\mathbf{E}$

$$-j\omega\mathbf{B}(r,m) = \hat{\mathbf{r}}\,[\,r^{-1}jmE_z + jkE_\theta] + \hat{\boldsymbol{\theta}}\,[-jkE_r - \partial_r E_z] + \hat{\mathbf{z}}\,[\,r^{-1}\partial_r(rE_\theta) - r^{-1}jmE_r\,] \qquad \text{(D.4.6)}$$

Things are pretty simple *except* for these two objects:

$\partial_r E_z$ $\qquad\qquad$ $\partial_r(rE_\theta)$

Basically we have to compute $\qquad\qquad$ $x = \beta' r$

$$\partial_r f_m = \beta'\,\partial_x f_m(x)$$

$$\partial_r(rE_\theta) = (\beta'\,\partial_x)\,[\,(x/\beta')\,h_m(x)\,] = \partial_x\,(x\,h_m(x))$$

Let's try to get Maple to compute these two objects. Let

$$df = \partial_x f_m(x)$$

and Maple says

```
restart; alias(I=I,j=sqrt(-1),J=BesselJ):
f := J(m,x)/J(m+1,xa) - J(m,x)/J(m-1,xa);
```

$$f := \frac{J(m,x)}{J(m+1,xa)} - \frac{J(m,x)}{J(m-1,xa)}$$

```
df := diff(f,x);
```

$$df := \frac{-J(m+1,x) + \frac{m\,J(m,x)}{x}}{J(m+1,xa)} - \frac{-J(m+1,x) + \frac{m\,J(m,x)}{x}}{J(m-1,xa)}$$

```
expand(%);
```

$$-\frac{J(m+1,x)}{J(m+1,xa)} + \frac{m\,J(m,x)}{J(m+1,xa)\,x} + \frac{J(m+1,x)}{J(m-1,xa)} - \frac{m\,J(m,x)}{J(m-1,xa)\,x}$$

The first and third term I recognize as + h (x)

```
h := J(m+1,x)/J(m+1,xa) - J(m-1,x)/J(m-1,xa);
```

$$h := \frac{J(m+1,x)}{J(m+1,xa)} - \frac{J(m-1,x)}{J(m-1,xa)}$$

The first item $\partial_x f_m(x)$ is pretty simple

$$f'(x) = \partial_x f_m(x) = J_m'(x)\,\left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right]$$

$$\partial_r f_m = \beta'\;J_m'(x)\,\left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right]$$

and this is just some other function of x and x_a for me to add to the list. The second item is

$$\partial_x (x\, h_m(x)) = x\, h'_m(x) + h_m(x)$$

and

$$h'_m(x) = \frac{J'_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J'_{m-1}(x)}{J_{m-1}(x_a)}$$

Now we then get

$$x\, h'_m(x) + h_m(x) = x\left[\frac{J'_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J'_{m-1}(x)}{J_{m-1}(x_a)}\right] + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}$$

$$= \frac{xJ'_{m+1}(x) + J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{xJ'_{m-1}(x) + J_{m-1}(x)}{J_{m-1}(x_a)}$$

I do know that from Schaum p 137 24.19 and 24.20 that

$$x\, J'_{m+1} = x\, J_m - (m+1)J_{m+1}$$

$$x\, J'_{m-1} = (m-1)J_{m-1} - x\, J_m$$

Then

$$\frac{xJ'_{m+1}(x) + J_{m+1}(x)}{J_{m+1}(x_a)} = \frac{x\, J_m(x) - (m+1)J_{m+1}(x) + J_{m+1}(x)}{J_{m+1}(x_a)} = \frac{x\, J_m(x) - mJ_{m+1}(x)}{J_{m+1}(x_a)}$$

$$\frac{xJ'_{m-1}(x) + J_{m-1}(x)}{J_{m-1}(x_a)} = \frac{(m-1)J_{m-1}(x) - x\, J_m(x) + J_{m-1}(x)}{J_{m-1}(x_a)} = \frac{mJ_{m-1}(x) - x\, J_m(x)}{J_{m-1}(x_a)}$$

Then

$$x\, h'_m(x) + h_m(x) = \frac{x\, J_m(x) - mJ_{m+1}(x)}{J_{m+1}(x_a)} - \frac{mJ_{m-1}(x) - x\, J_m(x)}{J_{m-1}(x_a)}$$

$$= x\left[\frac{J_m(x)}{J_{m+1}(x_a)} + \frac{J_m(x)}{J_{m-1}(x_a)}\right] - m\left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

$$= x\left[\frac{J_m(x)}{J_{m+1}(x_a)} + \frac{J_m(x)}{J_{m-1}(x_a)}\right] - m\, g_m$$

which is pretty simple. What single letter should I use for this sum. It is sort of f-Plus. Perhaps e. Here is Maple code to confirm the above. First just enter

```
restart; alias(I=I,j=sqrt(-1),J=BesselJ):
e :=  J(m,x)/J(m+1,xa) + J(m,x)/J(m-1,xa);
```

$$e := \frac{J(m,x)}{J(m+1,xa)} + \frac{J(m,x)}{J(m-1,xa)}$$

```
f := J(m,x)/J(m+1,xa) - J(m,x)/J(m-1,xa);
```

$$f := \frac{J(m,x)}{J(m+1,xa)} - \frac{J(m,x)}{J(m-1,xa)}$$

```
g := J(m+1,x)/J(m+1,xa) + J(m-1,x)/J(m-1,xa);
```

$$g := \frac{J(m+1,x)}{J(m+1,xa)} + \frac{J(m-1,x)}{J(m-1,xa)}$$

```
h := J(m+1,x)/J(m+1,xa) - J(m-1,x)/J(m-1,xa);
```

$$h := \frac{J(m+1,x)}{J(m+1,xa)} - \frac{J(m-1,x)}{J(m-1,xa)}$$

Then we want to show that

$$\partial_x (x\, h_m(x)) = x\, e_m - m\, g_m$$

```
> dxh := diff(x*h,x);
```

$$dxh := \frac{J(m+1,x)}{J(m+1,xa)} - \frac{J(m-1,x)}{J(m-1,xa)} + x\left(\frac{J(m,x) - \frac{(m+1)\,J(m+1,x)}{x}}{J(m+1,xa)} - \frac{-J(m,x) + \frac{(m-1)\,J(m-1,x)}{x}}{J(m-1,xa)}\right)$$

```
> dxh1 := x*e - m*g;
```

$$dxh1 := x\left(\frac{J(m,x)}{J(m+1,xa)} + \frac{J(m,x)}{J(m-1,xa)}\right) - m\left(\frac{J(m+1,x)}{J(m+1,xa)} + \frac{J(m-1,x)}{J(m-1,xa)}\right)$$

```
> dxh-dxh1: simplify(%);
```

$$0$$

So that was not so bad. Meanwhile

```
df := diff(f,x);
```

$$df := \frac{-J(m+1,x) + \frac{m\,J(m,x)}{x}}{J(m+1,xa)} - \frac{-J(m+1,x) + \frac{m\,J(m,x)}{x}}{J(m-1,xa)}$$

```
expand(%);
```

$$-\frac{J(m+1,x)}{J(m+1,xa)} + \frac{m\,J(m,x)}{J(m+1,xa)\,x} + \frac{J(m+1,x)}{J(m-1,xa)} - \frac{m\,J(m,x)}{J(m-1,xa)\,x}$$

The last line looks like

$$df = (m/x)\ f_m - J_{m+1}(x)\left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right]$$

$$= (m/x)\ f_m - \frac{J_{m+1}(x)}{J_m(x)}\, f_m = f_m\left[(m/x) - \frac{J_{m+1}(x)}{J_m(x)}\right]$$

Let's try to verify that:

```
df := diff(f,x);
```

$$df := \frac{-J(m+1,x)+\frac{m\,J(m,x)}{x}}{J(m+1,xa)} - \frac{-J(m+1,x)+\frac{m\,J(m,x)}{x}}{J(m-1,xa)}$$

```
dfe := expand(%);
```

$$dfe := -\frac{J(m+1,x)}{J(m+1,xa)} + \frac{m\,J(m,x)}{J(m+1,xa)\,x} + \frac{J(m+1,x)}{J(m-1,xa)} - \frac{m\,J(m,x)}{J(m-1,xa)\,x}$$

```
df1 := f * (m/x - J(m+1,x)/J(m,x));
```

$$df1 := \left(\frac{J(m,x)}{J(m+1,xa)} - \frac{J(m,x)}{J(m-1,xa)}\right)\left(\frac{m}{x} - \frac{J(m+1,x)}{J(m,x)}\right)$$

```
df1e :=expand(%);
```

$$df1e := -\frac{J(m+1,x)}{J(m+1,xa)} + \frac{m\,J(m,x)}{J(m+1,xa)\,x} + \frac{J(m+1,x)}{J(m-1,xa)} - \frac{m\,J(m,x)}{J(m-1,xa)\,x}$$

```
dfe-df1e;
```

$$0$$

Here then are my results: First,

$$\partial_x f_m(x) = f_m \left[(m/x) - \frac{J_{m+1}(x)}{J_m(x)} \right]$$

$$\partial_x (x\, h_m(x)) = x\, e_m - m\, g_m$$

Then

$$\partial_r f_m = \beta' \partial_x f_m(x) = \beta' f_m \left[(m/x) - \frac{J_{m+1}(x)}{J_m(x)} \right]$$

We are then going to get

$$\partial_r E_z = (1/4)\, \eta_m\, I\, R_{dc}\, (a\beta')\, \partial_r f_m$$

$$= (1/4)\, \eta_m\, I\, R_{dc}\, (a\beta')\, \beta' f_m \left[(m/x) - \frac{J_{m+1}(x)}{J_m(x)} \right]$$

$$= (1/4)\, \eta_m\, I\, R_{dc}\, (a)\, \beta'^2\, f_m \left[(m/x) - \frac{J_{m+1}(x)}{J_m(x)} \right] \qquad \dim(\text{RHS}) = \text{volts} = \text{wrong}$$

Next

$$\partial_r(rE_\theta) = \partial_r(r(1/4)\, \eta_m\, I\, R_{dc}\, (ak)\, h_m) = (1/4)\, \eta_m\, I\, R_{dc}\, (ak)\, \partial_r(r\, h_m)$$

$= (1/4)\,\eta_m\, I\, R_{dc}\ (ak)\ \partial_x(x\, h_m)$

$= (1/4)\,\eta_m\, I\, R_{dc}\ (ak)\ [\, x\, e_m - m\, g_m]$

To summarize, our two items needed are

$$\partial_r E_z = \ (1/4)\,\eta_m\, I\, R_{dc}\ (a)\ \beta'^2\ f_m\ [\ (m/x) - \frac{J_{m+1}(x)}{J_m(x)}\ ] \qquad \dim(\text{RHS}) = \text{volts} = \text{wrong!}$$

$$\partial_r(rE_\theta) = (1/4)\,\eta_m\, I\, R_{dc}\ (ak)\ [\, x\, e_m - m\, g_m]$$

We can then state our curl stuff:

$$\mathbf{B}(r,m) = (j/\omega)\{\ \hat{\mathbf{r}}\ [\ r^{-1}jmE_z + jkE_\theta] + \hat{\boldsymbol{\theta}}\ [-jkE_r - \partial_r E_z] + \hat{\mathbf{z}}\ [\ r^{-1}\partial_r(rE_\theta) - r^{-1}jmE_r\ ]\ \ \} \qquad \text{(D.4.6)}$$

$B_r(r,m) = (j/\omega)[\ r^{-1}jmE_z + jkE_\theta]$

$= (j/\omega)[\ r^{-1}jm\ (1/4)\,\eta_m\, I\, R_{dc}\ (a\beta')\ f_m + jk(1/4)\,\eta_m\, I\, R_{dc}\ (ak)\ h_m]$

$= (1/4)\,\eta_m\, I\, R_{dc}\ (j/\omega)\ [\ r^{-1}jm\ (a\beta')\ f_m + jk\ (ak)\ h_m]$

$= (1/4)\,\eta_m\, I\, R_{dc}\ (j/\omega)\ ja\ [\ r^{-1}m\ (\beta')\ f_m + k^2 h_m]$

$= -\ (1/4)\,\eta_m\, I\, R_{dc}\ (a/\omega)\ [\ r^{-1}m\ (\beta')\ f_m + k^2\ h_m]$

which is a pretty simple result really! Check dimensions

$\dim(\text{RHS}) = \text{volts/m} * \text{m-sec} * 1/\text{m}^2 = \text{volts-sec/m}^2$ OK

Let's try the next one

$B_\theta(r,m) = (j/\omega)[\ -jkE_r - \partial_r E_z\ ]$

$$= (j/\omega)[\ -jk(j/4)\,\eta_m\, I\, R_{dc}\ (ak)\ g_m - (1/4)\,\eta_m\, I\, R_{dc}\ (a)\ \beta'^2\ f_m\ [\ (m/x) - \frac{J_{m+1}(x)}{J_m(x)}\ ]\ ]$$

$$= (1/4)(j/\omega)\,\eta_m\, I\, R_{dc}\ \{\ -jk(j)\ (ak)\ g_m - \ (a)\ \beta'^2\ f_m\ [\ (m/x) - \frac{J_{m+1}(x)}{J_m(x)}\ ]\ \}$$

$$= (1/4)(j/\omega)\,\eta_m\, I\, R_{dc}\ \{k\ (ak)\ g_m - \ (a)\ \beta'^2\ f_m\ [\ (m/x) - \frac{J_{m+1}(x)}{J_m(x)}\ ]\ \}$$

$$= (1/4)(j/\omega)\,\eta_m\, I\, R_{dc}\ \{ak^2\ g_m - \ a\ \beta'^2\ f_m\ [\ (m/x) - \frac{J_{m+1}(x)}{J_m(x)}\ ]\ \}$$

$$= (j/4)(a/\omega)\,\eta_m\, I\, R_{dc}\ \{k^2\ g_m - \ \beta'^2 f_m\ [\ (m/x) - J_{m+1}(x)/J_m(x)]\ \}$$

This is a little messier, but still not too bad. Finally,

$$B_z(r,m) = (j/\omega) \left[r^{-1}\partial_r(rE_\theta) - r^{-1}jmE_r \right]$$

$$= (j/\omega) \left[r^{-1}(1/4)\,\eta_m I R_{dc}\,(ak) [x\, e_m - m\, g_m] - r^{-1}jm(j/4)\,\eta_m I R_{dc}\,(ak)\, g_m \right]$$

$$= (1/4)(j/\omega)\,\eta_m I R_{dc} \left[r^{-1}\,(ak) [x\, e_m - m\, g_m] - r^{-1}jm\,(j)\,(ak)\, g_m \right]$$

$$= (j/4)(a/\omega)\,\eta_m I R_{dc} \left[r^{-1}\,(k) [x\, e_m - m\, g_m] - r^{-1}jm\,(j)\,(k)\, g_m \right]$$

$$= (j/4)(a/\omega)\,\eta_m I R_{dc} k r^{-1} \left[[x\, e_m - m\, g_m] + m\, g_m \right]$$

$$= (j/4)(a/\omega)\,\eta_m I R_{dc} k r^{-1} \left[x\, e_m - m\, g_m + m\, g_m \right]$$

$$= (j/4)(a/\omega)\,\eta_m I R_{dc} k\, r^{-1} \left[x\, e_m \right]$$

with a lucky cancellation at the very end! I will of course have to check all these things somehow.

Results so far are

$$B_r(r,m) = -(1/4)(a/\omega)\,\eta_m I R_{dc} \{ r^{-1} m\, (\beta')\, f_m + k^2\, h_m \}$$

$$B_\theta(r,m) = (j/4)(a/\omega)\,\eta_m I R_{dc} \{ k^2\, g_m - \beta'^2 f_m [(m/x) - J_{m+1}(x)/J_m(x)] \}$$

$$B_z(r,m) = (j/4)(a/\omega)\,\eta_m I R_{dc} \{ k\, r^{-1}\, x\, e_m \} = (j/4)(a/\omega)\,\eta_m I R_{dc} \{ k\, \beta'\, e_m \}$$

I will of course have to check these results carefully, but the presentation is compact! This is very much better than what I have in Appendix D right now. Dimensions all OK.

Phase 2. Let's assume the above are correct and see if I can make Maple verify them! To simplify things, lets just delete the following factor which is common to all E_i and all B_i fields

$$(1/4)\,\eta_m I R_{dc}\, a$$

Then we can say

$$E_z(r,m) = \beta' f_m$$
$$E_r(r,m) = j k g_m$$
$$E_\theta(r,m) = k h_m$$

$$-j\omega B_r(r,m) = j (r^{-1} m\, (\beta')\, f_m + k^2\, h_m)$$
$$-j\omega B_\theta(r,m) = (k^2 g_m - \beta'^2 f_m [(m/x) - J_{m+1}(x)/J_m(x)])$$
$$-j\omega B_z(r,m) = (k\, r^{-1}\, x\, e_m) = k r^{-1} \beta' r\, e_m = k\, \beta'\, e_m$$

These six fields should satisfy the curl E equation, and it is now our task to check

$-j\omega\ \mathbf{B} = \text{curl}\ \mathbf{E}$

$\hat{\mathbf{r}}\ [\ r^{-1}jmE_z + jkE_\theta] + \hat{\boldsymbol{\theta}}\ [-jkE_r - \partial_r E_z] + \hat{\mathbf{z}}\ [\ r^{-1}\partial_r(rE_\theta) - r^{-1}jmE_r\ ]$

So it is my task to show these three things:

$-j\omega B_r(r,m) = [\text{curl}\ \mathbf{E}]_r$
$-j\omega B_\theta(r,m) = [\text{curl}\ \mathbf{E}]_\theta$
$-j\omega B_z(r,m) = [\text{curl}\ \mathbf{E}]_z$

In Maple lets use

$-j\omega B_r$ = mjwBr and so on

Then the equations are

mjwBr = curlEr
mjwBth = curlEth
mjwBz = curlEz