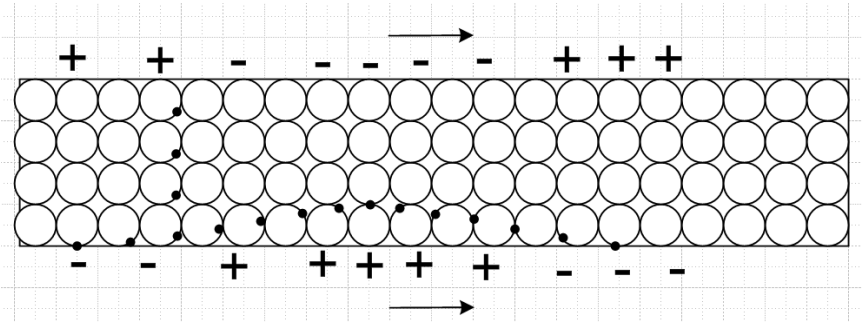
Elliptical orbits of electrons in a round wire PhL 12.8.13

I think I am happy with tiny elliptical orbits for individual charges. Consider a monstrous number of electrons in the surface charge layer strung along a line in the z direction that is λ long. The "undulation" with period T of this charge distribution is known, and arises from *phase differences* on those ellipses.



This picture shows how that might happen. You could "picture" the surface charge *function* as moving left to right in this drawing, OR you could imagine it as moving "radially" in the wire. These phased movements of individual electrons cause macroscopic currents J which we study in Appendix D.

It is very easy to have misconceptions in E&M, especially if you are "rusty" as I am.

1. Question: As a wave moves down a round wire, the charge pattern seems to move with the wave. Does this mean that the surface charge is simply moving down the wire longitudinally?

This brings up many strange questions.

- If the charge density n(φ,z) simply slides along in the z direction, then does that mean it does not move in the φ direction at all?
- Maybe the charge density is "fed" by the radial currents in the wire, and that is what makes it change, and then it moves neither in the φ *nor* the z direction, but only in the r direction!

This is a subtle business. If you look at an ocean wave, the wave moves in the z direction just as our charge pattern moves in the z direction. But an individual droplet of water does NOT "simply move in the z direction". In fact, a droplet of water goes in some elliptical path and never makes progress at all in the z direction.

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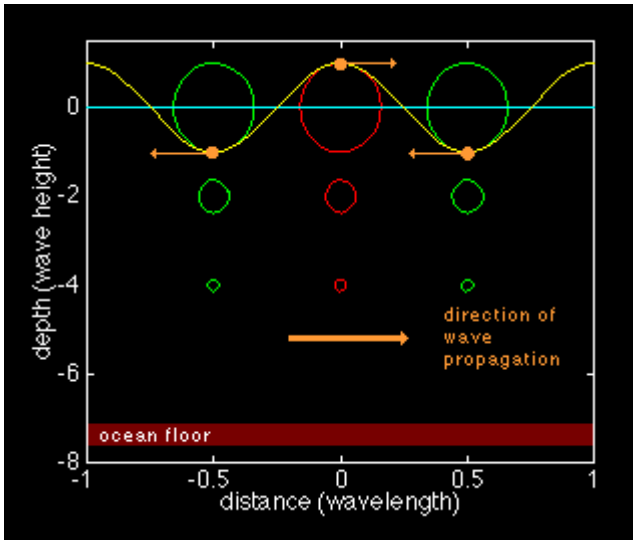
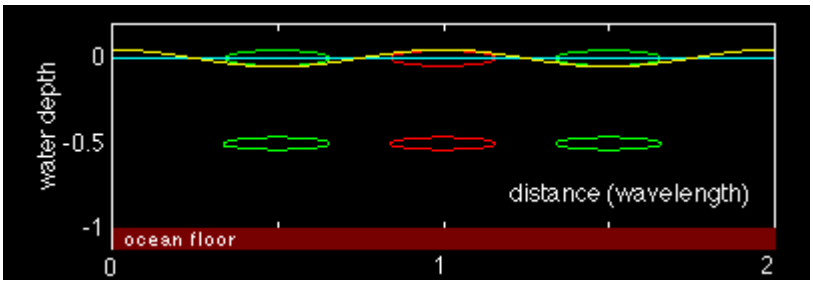
http://labman.phys.utk.edu/phys221/modules/m11/Water_waves.htm

The water molecules of a **deep-water wave** move in a circular orbit. The diameter of the orbit decreases with the distance from the surface. The motion is felt down to a distance of approximately one wavelength, where the wave's energy becomes negligible.



The orbits of the molecules of **shallow-water waves** are more elliptical.



The change from deep to shallow water waves occurs when the depth of the water, d, becomes less than one half of the wavelength of the wave, λ. When d is much greater than λ/2 we have a **deep-water wave or a short wave**. When d is much less than λ/2 we have a **shallow-water wave or a long wave**.

So in this more physical-world example, a chunk of water does not move along with the wave, and that makes it seem possible that a "piece of surface charge" does not simply move in the z direction in a round wire. Analogies are strong teachers, but doubtless can also be misleading. Here we have water with a surface compared with a conductor and its surface.

If we were to look at a tiny volume of conduction electrons *inside* a round wire, I think we would find that that chunk of electrons in fact goes in an elliptical pattern. We would have to "tag" the electrons somehow to watch them. Inside the wire, we have fields E_z, E_r and E_φ none of which is zero, and being complex each has its own phase. In the time domain, we have for a given partial wave m,

$$E_z(r,\varphi,z,t) = \text{Re}\ \{\ e^{j(\omega t - kz)}\ e^{im\varphi}\ e^{j\varphi_z}\ E_z(r,m)\ \} = \cos(\omega t - kz - m\varphi - \varphi_z) E_z(r,m)$$

So the three field components are doing this (the E_i are real)

$$E_z(r,\varphi,z,t) = \cos[\omega t - kz - m\varphi - \varphi_z(r,m)]\ E_z(r,m)$$

$E_z(r,\varphi,z,t) = \cos[\omega t - kz - m\varphi - \varphi_z(r,m)]\ E_z(r,m)$
$E_\varphi(r,\varphi,z,t) = \cos[\omega t - kz - m\varphi - \varphi_\varphi(r,m)]\ E_\varphi(r,m)$

Even though the metal is electrically neutral, the positive ion lattice is heavy and does not move, but an electron with charge q does feel the force **F** = q**E** (ignoring B) and moves in response, and this electron is in our "droplet" of electrons. The physical electron position is determined by **F** = m**a** so **a** = q**E**/m and then the fields shown above are causing an acceleration of the electron. So then

$a_z(r,\varphi,z,t) = \cos[\omega t - kz - m\varphi - \varphi_z(r,m)]\ E_z(r,m)(q/m)$
$a_r(r,\varphi,z,t) = \cos[\omega t - kz - m\varphi - \varphi_r(r,m)]\ E_r(r,m)\ (q/m)$
$a_\varphi(r,\varphi,z,t) = \cos[\omega t - kz - m\varphi - \varphi_\varphi(r,m)]\ E_\varphi(r,m)\ (q/m)$

If we think of a tiny Cartesian cube with axes z,r,φ (hatted as unit vectors, away from the axis r = 0) then we can do Cartesian mechanics and it would seem then that

$x_z(r,\varphi,z,t) = -\cos[\omega t - kz - m\varphi - \varphi_z(r,m)]\ E_z(r,m)(q/\omega^2 m)$
$x_r(r,\varphi,z,t) = -\cos[\omega t - kz - m\varphi - \varphi_r(r,m)]\ E_r(r,m)\ (q/\omega^2 m)$
$x_\varphi(r,\varphi,z,t) = -\cos[\omega t - kz - m\varphi - \varphi_\varphi(r,m)]\ E_\varphi(r,m)\ (q/\omega^2 m)$

At a particular location r,φ,z in the wire this has the form (I use x,y,z now for these little coords)

$x(t) = \cos[\omega t - a]\ A$
$y(t) = \cos[\omega t - b]\ B$
$z(t) = \cos[\omega t - c]\ C$

If I just plot this in Maple, it is looks like a tilted ellipse in 3D space. We know the orbit is closed when we hit one period and then it just repeats. But I was wondering (curiosity killed the cat) what the equation of this orbit might be? Is it in fact just an ellipse? // I wasted all morning trying to answer this question, see separate doc, it was inconclusive. All I can say is "the curve is ellipse-like". That is good enough for today in this doc!

[This led to a 2 1/2 day digression on geometry and calculus, all of which is now stored in a little folder in my geometry area. The conclusion is that the electron orbits *really are* ellipses, but the geometry is amazingly complicated!!! The ellipse has certain major and minor axes and a certain orientation in 3D space, and all these things are very complicated functions of the 6 parameters shown.]

Conclusion #1: Inside a round wire, if you were to "tag" a small volume of conduction electrons, you would see that small volume making an ellipse-like [elliptical] orbit in 3D. The conduction electrons don't "go anywhere" non-locally, they stay in their neighborhood as the wave passes by.

I have ignored the magnetic part of the force, assuming it is down by (v/c) somehow. This site argues that the "drift speed" of electrons is tiny, on the order of 4 mm/sec or less.

http://hyperphysics.phy-astr.gsu.edu/hbase/electric/ohmmic.html

Probably the orbit size is very small, but I have not tried to calculate it. [I do right below!]

Well, suppose you are at a frequency of 100 Hz so a half period time is .0005 seconds. During this time, an electron with a drift velocity of 4 mm/sec travels $x = 5 \times 10^{-3}$ sec $* 4 \times 10^{-3}$m $= 20 \times 10^{-6}$ m = 20 microns. So in a power transmission line running at 100 Hz, an oscillating free electron might move this far in its "elliptical motion" extent. At this frequency, that extent would be all in the z direction. At any RF frequency, can see that the ellipse extent is on the order of $x = 20 \times 10^{-6} * (100/f) = 20 \times 10^{-4} / f$ meters

$x = 2x\ 10^{-3}\ (1/f)$ meters

For example at 1 MHz motion is $x = 2 \times 10^{-9}$ me = 2 nanometers. So yes, orbit is small!

This conclusion deals with the *inside* of the wire. It says that you don't really have a bulk flow down the wire of conduction electrons in an AC situation. Any given electron jiggles back and forth in each direction z, r, and φ, but stays put in a global sense.

Resuming here on 12/11/13 after geometry digression noted above.

So what happens with the surface charge?

I model this a bit in lines doc Appendix E. The idea there is $\mathbf{J} = \sigma\mathbf{E}$ - D grad ρ modification of the bulk Ohm's Law. Current is like heat flow, ρ like temperature, gradient in ρ causes some J. Maybe $\sigma\mathbf{E}$ is like convection heat flow added in.

If you look normal to the surface, you get $J_n = \sigma E_n - D\partial_n\rho$. No current can flow through the surface, so you end up with $0 = \sigma E_n - D\partial_n\rho$ and then $E_n = D\partial_n\rho/\sigma$ as your normal field at the surface. When you add in the idea that div $E = \rho/\varepsilon$ you get in this simple geometry $\partial_n E_n = \rho/\varepsilon$ so we have

$E_n = (D/\sigma)\partial_n\rho \qquad \partial_n E_n = \rho/\varepsilon$

$\Rightarrow \qquad \rho/\varepsilon = \partial_n E_n = (D/\sigma)\partial_n{}^2\rho \qquad \Rightarrow \qquad \partial_x{}^2\rho = (\sigma/\varepsilon D)\rho = (1/\lambda_D{}^2)\,\rho$

$\Rightarrow \qquad \rho = \exp(-x/\lambda_D) \qquad\qquad \lambda_D$ = "depth of the surface charge"

This then causes ρ to have an exponential decay profile from the surface into the metal and the depth is that constant λ_D, more or less. This layer of free charge just inside the metal surface is what we call "the surface charge". If we just think of the surface charge layer as having thickness λ_D, then

$n\ (C/m^2) = \rho(C/m^3)\ \lambda_D\ (m)$ = surface charge density ok to here

If there is a tangential E field just inside the surface, it seems to me it would push that surface charge along tangentially. That is why in electrostatics, E is normal to the surface and there is no tangential E field because things are static. I don't think the D grad ρ term has any effect on $J_t = \sigma E_t$ for tangential E field and tangential current flow. ok

In a non-static situation, we have a certain radial E field (here a normal E field E_n) which is time varying. But we still have $J_n(r=a) = 0$ at the surface because no charge can leave the metal. Then I guess we have this equation

$$\partial_{\mathbf{n}} E_{\mathbf{n}}(r,t) = \rho(r,t)/\varepsilon \quad = n(t)/\lambda_{\mathbf{D}}$$

This just says that the $E_{\mathbf{n}}$ normal gradient and the surface charge are locked together. This fact applies only inside the surface charge layer where $\rho \neq 0$.

So you really can think of $\partial_{\mathbf{n}} E_{\mathbf{n}}$ as "pumping the surface charge" since it is locked to the surface charge in this way.

What about div $J_{\mathbf{free}} = -\partial_{\mathbf{t}} \rho_{\mathbf{free}}$ and div $J_{\mathbf{bound}} = -\partial_{\mathbf{t}} \rho_{\mathbf{bound}}$? If both sides of the boundary have the same ε, there is no bound charge anywhere. When we talk about J, we normally mean $J_{\mathbf{bound}}$ as in Ohm's Law. So we have div J = 0 and the surface charge plays no role in this. On the other hand, we seem to have

$$\text{div } J_{\mathbf{free}} = -\partial_{\mathbf{t}} \rho_{\mathbf{free}} = -\partial_{\mathbf{t}} \rho(t)$$

so there must be some free charge current near the surface. This is what does the pumping.

Misconception: I thought that $E_{\mathbf{r}}$ was pumping the surface charge. Well, below the surface this does make some $J_{\mathbf{r}}$ bulk radial current flowing perhaps toward the surface. Where do those electrons go, there is no way out, they have to go into n(t) it seems to me. I am missing something, as usual. This general subject comes up in Section 1.5 of lines.