Extending the cpbc for $\sigma_d \neq 0$ PhL 6.2.14

Extending the Charge Pump Boundary Condition of Appendix D. This stuff has been installed as section D.9 (d).

Question: If I allow the dielectric to conduct, how does that change the cpbc? It would then read

$$J_r(a-\alpha,\theta) - J_r(a+\alpha,\theta) = (j\omega)\, n(\theta)$$

or

$$\sigma\, E_r(a-\alpha,\theta) - \sigma_d E_r(a+\alpha,\theta) = (j\omega)\, n(\theta) \qquad \text{// this is div } J = -j\omega\rho$$

But we know from (1.1.47) that [$\hat{n}$ points into medium 1 which for me is the dielectric]

$$[\varepsilon_1 E_{n1} - \varepsilon_2 E_{n2}] = n_{free} \qquad \text{// this is continuity of } D_n$$

or

$$[\varepsilon_d E_{r1} - \varepsilon_0 E_{r2}] = n_{free}$$

or

$$[\varepsilon_d\, E_r(a+\alpha,\theta) - \varepsilon_0\, E_r(a-\alpha,\theta)] = n_{free}$$

So how does this relate to the cpbc?? It must be that $n_{free} = n(\theta)$ = the physical surface charge. so

$$\sigma\, E_r(a-\alpha,\theta) - \sigma_d E_r(a+\alpha,\theta) = (j\omega)\, n(\theta)$$

$$\varepsilon_d\, E_r(a+\alpha,\theta) - \varepsilon_0\, E_r(a-\alpha,\theta) = n(\theta)$$

But should not these two equations be the same? No! They both appear below (1.1.47). You can eliminate $n(\theta)$ and end up with

$$\xi_1 E_{n1} = \xi_2 E_{n2}$$

or

$$(\varepsilon_1 + \sigma_1/j\omega)\, E_{n1} = (\varepsilon_2 + \sigma_2/j\omega)\, E_{n2}$$

or

$$(\varepsilon_d + \sigma_d/j\omega)\, E_{nd} = (\varepsilon_0 + \sigma/j\omega)\, E_n$$

or

$$(\varepsilon_d + \sigma_d/j\omega)\, E_{rd} \approx (\sigma/j\omega)\, E_r$$

which then relates the two fields so

$$(\varepsilon_d + \sigma_d/j\omega)\, E_r(a+\alpha,\theta) \approx (\sigma/j\omega)\, E_r(a-\alpha,\theta)$$

But this provides another "connection" between inside and outside that *I forgot about*! Ouch!!!! I cannot ignore this thread now, but it is 10AM pre Torrey.

Summary: there are two independent conditions here relating inside and outside E_r fields. I can take any pair of these three equations

$\sigma\, E_{\mathbf{r}}(a-\alpha,\theta) \;-\; \sigma_{\mathbf{d}} E_{\mathbf{r}}(a+\alpha,\theta) \;= (j\omega)\, n(\theta)$ // modified cpbc from div **J** = -jωρ

$\varepsilon_{\mathbf{d}}\, E_{\mathbf{r}}(a+\alpha,\theta) \;-\; \varepsilon_0\, E_{\mathbf{r}}(a-\alpha,\theta) \;=\; n(\theta)$ // div **D** = ρ straddle

$(\varepsilon_{\mathbf{d}} + \sigma_{\mathbf{d}}/j\omega)\, E_{\mathbf{r}}(a+\alpha,\theta) \approx\; (\sigma/j\omega)\, E_{\mathbf{r}}(a-\alpha,\theta)$ // $\xi_1 E_{\mathbf{n1}} = \xi_2 E_{\mathbf{n2}}$

NOTE: if ρ= 0, then div **E** = 0 is the same as div **J** = 0, but this is not true otherwise!!!

For a vacuum dielectric with $\sigma_{\mathbf{d}} = 0$ these three conditions read

$\sigma\, E_{\mathbf{r}}(a-\alpha,\theta) \;= (j\omega)\, n(\theta)$ // cpbc from div **J** = -jωρ

$\varepsilon_{\mathbf{d}}\, E_{\mathbf{r}}(a+\alpha,\theta) \;-\; \varepsilon_0\, E_{\mathbf{r}}(a-\alpha,\theta) \;=\; n(\theta)$ // div **D** = ρ (**)

$(\varepsilon_{\mathbf{d}})\, E_{\mathbf{r}}(a+\alpha,\theta) \approx\; (\sigma/j\omega)\, E_{\mathbf{r}}(a-\alpha,\theta)$ // $\xi_1 E_{\mathbf{n1}} = \xi_2 E_{\mathbf{n2}}$ (*)

The last one at least does say the field inside is "small" since σ is large. In the capacitor model of the transmission line, I *assume* that $E_{\mathbf{r}}$(inside) ≡ 0 and that is how I *get* n(θ). If I took (*) strictly, then I would be unable to solve the exterior problem using the capacitor method. Having ignored (*) to get n(θ) as an approximation, I suppose (*) is still reasonable if you want to know $E_{\mathbf{r}}$ inside.

$E_{\mathbf{r}}(a-\alpha,\theta) \approx [j\omega\, \varepsilon_{\mathbf{d}}/\sigma]\, E_{\mathbf{r}}(a+\alpha,\theta)$

Now, to get n(θ) outside I had to say this:

$n(\theta) = \varepsilon_{\mathbf{d}} E_{\mathbf{r}}(a+\alpha,\theta)$ // from (**)

But then (*) reads

$n(\theta) \approx\; (\sigma/j\omega)\, E_{\mathbf{r}}(a-\alpha,\theta)$

and this is the same as the cpbc. So if I decide to compute n(θ) as I do assuming $E_{\mathbf{r}} \approx 0$ inside, then I don't really have two independent conditions, I have only one which is the cpbc.

Now let's go back to the original question.

********** continuing on June 1 **********

Now let's go back to the original question which was:

Question: If I allow the dielectric to conduct, how does that change the cpbc?

In this case we have these three boundary conditions :

1 $\sigma E_r(a-\alpha,\theta) - \sigma_d E_r(a+\alpha,\theta) = (j\omega)\, n(\theta)$ // modified cpbc from div **J** = -jωρ

2 $\varepsilon_d E_r(a+\alpha,\theta) - \varepsilon_0 E_r(a-\alpha,\theta) = n(\theta)$ // div **D** = ρ straddle

3 $(\varepsilon_d + \sigma_d/j\omega) E_r(a+\alpha,\theta) \approx (\sigma/j\omega) E_r(a-\alpha,\theta)$ // $\xi_1 E_{n1} = \xi_2 E_{n2}$

Claim A: only two of these three BC's are independent.

Proof: If we take -1 divided by jω we get

4 $(\sigma_d/j\omega)E_r(a+\alpha,\theta) - (\sigma/j\omega) E_r(a-\alpha,\theta) = - n(\theta)$

Now add this to 2 to get

$(\varepsilon_d + \sigma_d/j\omega) E_r(a+\alpha,\theta) - (\sigma/j\omega) E_r(a-\alpha,\theta) = 0$

but this is BC#3, so only two are independent. I then have 2 equations in 3 unknowns which are the functions $E_r(a-\alpha,\theta)$, $_dE_r(a+\alpha,\theta)$ and $n(\theta)$.

Claim B. When I solve the capacitor problem, I am really using #2 with the approximation that $E_r(a-\alpha,\theta) \approx 0$ inside the conductor. This then gives a result for $n(\theta)$. What I am really assuming there is that $E_r(a-\alpha,\theta) << E_r(a+\alpha,\theta)$ assuming ε_d and ε_0 are about the same. This then "uses up" equation #2, and I have only one other equation left.

It is unclear how to proceed from this point in order to answer the question : how is the cpbc altered by $\sigma_d \neq 0$. I cannot simply drop the term $\sigma_d E_r(a+\alpha,\theta)$ in #1 without some justification. The problem with that is that, although $E_r(a-\alpha,\theta)$ is very small, σ is very large, so the size of product $\sigma E_r(a-\alpha,\theta)$ is hazy.

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I am *expecting* involvement of this equation somehow, where n_s is the surface charge, and n_c is that other thing I discuss in Section 1.5 (c):

$n_s(\theta) = (\varepsilon_d/\xi_d)\, n_c(\theta)$

The n_s is what appears in the capacitor problem, so my three BC equations are

1 $\sigma E_r(a-\alpha,\theta) - \sigma_d E_r(a+\alpha,\theta) = (j\omega)\, n_s(\theta)$ // modified cpbc from div **J** = -jωρ

2 $\varepsilon_d E_r(a+\alpha,\theta) - \varepsilon_0 E_r(a-\alpha,\theta) = n_s(\theta)$ // div **D** = ρ straddle

3 $\xi_d E_r(a+\alpha,\theta) \approx \xi E_r(a-\alpha,\theta)$ // $\xi_1 E_{n1} = \xi_2 E_{n2}$

So what might I do here? Suppose I eliminate n_s in favor of n_c :

1 $\sigma E_r(a-\alpha,\theta) - \sigma_d E_r(a+\alpha,\theta) = (j\omega)\, (\varepsilon_d/\xi_d)\, n_c(\theta)$

2 $\varepsilon_d E_r(a+\alpha,\theta) - \varepsilon_0 E_r(a-\alpha,\theta) = (\varepsilon_d/\xi_d)\, n_c(\theta)$

3 $\xi_d E_r(a+\alpha,\theta) \approx \xi E_r(a-\alpha,\theta)$

How does this help? Rewrite as

1 $\sigma\ (\xi_d/\varepsilon_d)\ E_r(a-\alpha,\theta)\ -\ \sigma_d\ (\xi_d/\varepsilon_d)\ E_r(a+\alpha,\theta)\ = (j\omega)\ n_c(\theta)$
2 $\xi_d\ E_r(a+\alpha,\theta)\ -\ (\xi_d/\varepsilon_d)\ \varepsilon_0\ E_r(a-\alpha,\theta)\ =\ n_c(\theta)$
3 $\xi_d E_r(a+\alpha,\theta) \approx \xi\ E_r(a-\alpha,\theta)$

Now use #3 to eliminate $E_r(a+\alpha,\theta)$ in #1:

$\sigma\ (\xi_d/\varepsilon_d)\ E_r(a-\alpha,\theta)\ -\ \sigma_d\ (\xi_d/\varepsilon_d)\ E_r(a+\alpha,\theta)\ = (j\omega)\ n_c(\theta)$
or
$\sigma\ (\xi_d/\varepsilon_d)\ E_r(a-\alpha,\theta)\ -\ \sigma_d\ (\xi/\varepsilon_d)\ E_r(a-\alpha,\theta)\ = (j\omega)\ n_c(\theta)$
or
$[\ \sigma\ (\xi_d/\varepsilon_d) -\ \sigma_d\ (\xi/\varepsilon_d)\ ]\ E_r(a-\alpha,\theta)\ = (j\omega)\ n_c(\theta)$
or
$[\ \sigma\ \xi_d -\ \sigma_d\ \xi\ ]/\varepsilon_d\ E_r(a-\alpha,\theta)\ = (j\omega)\ n_c(\theta)$
or
$[\ \sigma\ (\varepsilon_d + \sigma_d/j\omega) -\ \sigma_d\ (\varepsilon_0 + \sigma/j\omega)\ ]/\varepsilon_d\ E_r(a-\alpha,\theta)\ = (j\omega)\ n_c(\theta)$
or
$[\ (\sigma\ \varepsilon_d - \sigma_d\varepsilon_0) + (\sigma\sigma_d-\sigma\sigma_d)/j\omega\ ]/\varepsilon_d\ E_r(a-\alpha,\theta)\ = (j\omega)\ n_c(\theta)$
or
$[\ (\sigma\ \varepsilon_d - \sigma_d\varepsilon_0)]/\varepsilon_d\ E_r(a-\alpha,\theta)\ = (j\omega)\ n_c(\theta)$
or
$[\ (\sigma - \sigma_d(\varepsilon_0/\varepsilon_d)\ ]\ E_r(a-\alpha,\theta)\ = (j\omega)\ n_c(\theta)$

Now finally: since the two ε are the same order, we drop the second term to get

$\sigma\ E_r(a-\alpha,\theta)\ = (j\omega)\ n_c(\theta)$

and THIS then is the modified cpbc ! This what I was hoping for. The main idea is the cancellation shown in red above.

Answer to the Question: the modified version of the cpbc is this

$\sigma\ E_r(a-\alpha,\theta)\ = (j\omega)\ n_c(\theta)$
or
$\sigma\ E_r(a-\alpha,\theta)\ = (j\omega)\ (\xi_d/\varepsilon_d)\ n_s(\theta)$
or
$\sigma\ E_r(a-\alpha,\theta)\ = (j\omega)\ ([(\varepsilon_d + \sigma_d/j\omega)]/\varepsilon_d)\ n_s(\theta)$
or
$\sigma\ E_r(a-\alpha,\theta)\ = (j\omega)\ (1 + \sigma_d/j\omega\varepsilon_d\ )\ n_s(\theta)$

We can then ignore the $\sigma_d/j\omega\varepsilon_d$ term here as long as

$\sigma_d/j\omega\varepsilon_d \ll 1$

We already know that $\omega\varepsilon \ll \sigma$ for $\omega < 10^{18}$, but this new condition is different. We have

$\sigma_d \ll \omega\varepsilon_d$

At very low ω this won't be realized. Write as'

$\sigma_d \ll 2\pi f\, \varepsilon_0 = 6.28 * 8.8541877 \times 10^{-12} * f \approx 54 * 10^{-12} \approx 10^{-10} f$?

So if f = 1 Hz, you only need $\sigma_d \ll 10^{-10}$ to be OK.

But when we use σ_{eff}, this probably *won't* be OK and the factor $(\xi_d/\varepsilon_d)\, n_s(\theta)$ must be maintained. So everywhere you have a charge type factor, you now must add (ξ_d/ε_d).

So how does that play out in the Appendix D results??? Take it in steps please.

1. We have to make these two replacements:

$n(\theta) \rightarrow (\xi_d/\varepsilon_d)\, n(\theta) \equiv n'(\theta)$

$N_m \rightarrow (\xi_d/\varepsilon_d)\, N_m \equiv N'_m$

2. The coefficient solutions then become

$$a_m = (j\omega/2\sigma)\, 2m\, N_m \frac{1}{J_{m-1}(x_a)} \rightarrow (\xi_d/\varepsilon_d)\, (j\omega/2\sigma)\, 2m\, N_m \frac{1}{J_{m-1}(x_a)}$$

$$= (j\omega/2\sigma)\, 2m\, N_m' \frac{1}{J_{m-1}(x_a)} = a'_m$$

and similarly for K_m,

$$\frac{K_m}{2} = (j\omega/2\sigma)\, N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] \rightarrow (\xi_d/\varepsilon_d)\, (j\omega/2\sigma)\, N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right]$$

$$= (j\omega/2\sigma)\, N_m' \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] = K'_m/2$$

$$\frac{K_0}{2} = (j\omega/\sigma)\, N_0 \frac{1}{J_1(x_a)} \rightarrow (\xi_d/\varepsilon_d)\, (j\omega/\sigma)\, N_0 \frac{1}{J_1(x_a)} = (j\omega/\sigma)\, N_0' \frac{1}{J_1(x_a)}$$

The coefficients get larger! I have just renamed them with a prime for the "new" situation.

3. The normalization condition (D.2.31) is then

$$(\xi_d/\varepsilon_d)N_0 = (\beta_d/2\pi\omega a)\, I' = N_0'$$

where I' is the current in the new situation. So,

or

$$(\beta_d/2\pi\omega a)\, I' = (\xi_d/\varepsilon_d)N_0$$

Suppose we start with $\sigma_d = 0$ with some I and some N_0 moment. Ignoring the change in β_d, you would argue that I "increases" due to this factor (ξ_d/ε_d), since N_0 does not change. That is, I' > I.

4. The new E_z field is this

$$E'_z(r,m) = -j(\beta'/\beta_d)\frac{K'_m}{2} J_m(x) = -j(\beta'/\beta_d)\,(j\omega/2\sigma)\, N_m' \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] J_m(x)$$

$$= -j(\beta'/\beta_d)\,(j\omega/2\sigma)\,(N_m'/N_0')\, N'_0 \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] J_m(x)$$

$$= -j(\beta'/\beta_d)\,(j\omega/2\sigma)\,\eta_m\, N'_0 \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] J_m(x)$$

Now use the fact that

$$(\beta_d/2\pi\omega a)\, I' = N_0'$$

to end up with

$$E'_z(r,m) = -j(\beta'/\beta_d)\,(j\omega/2\sigma)\,\eta_m\,(\beta_d/2\pi\omega a)\, I' \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] J_m(x)$$

$$= (\beta')\,(1/4\sigma)\,\eta_m\,(1/\pi a)\, I' \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] J_m(x)$$

$$= (1/4)\,\eta_m\, I'\, R_{dc}\ (a\beta') \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right]$$

Now $E'_z(r,m)$ is different for two reasons. First, because I' is different from the old I. Second, β_d is different.