Exterior Solution PhL 12.6.13

(12/12/13) In my original work here, I assumed you could have m ≠ 0 partial waves in the coaxial cable picture where r = ∞ is part of the picture. I later realized this is wrong as shown in pictures below. In the original work I ended up with a contraction. I had these 3 "boundary conditions"

$$\xi\, E_r(a,m) = \xi_d\, E_{rd}(a,m)$$
$$(-j\sigma/\omega) E_r(a,m) = \varepsilon_d\, E_{rd}(a,m)$$
$$(-j\sigma/\omega)\,(j\omega/\sigma)\, N_m = \varepsilon_d\, E_{rd}(a,m)$$

where "d" means dielectric or exterior solution. For the interior values I had

$$E_{zd}(a,m) = -\,j\,(\beta'/\beta_d)\,\frac{K_m}{2}\,J_m(x) \qquad \text{where} \quad \frac{K_m}{2} = (j\omega/2\sigma)\,N_m\left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right]$$
$$E_{rd}(a,m) = N_m/\varepsilon_d$$
$$jE_{\phi d}(a,m) = 0$$

where the last is questionable but OK for the moment. But for the exterior values I had

$$E_{zd}(r,m) = B_{zm} r^{-m}$$
$$E_{rd}(r,m) = -(1/2)(m-1)^{-1}\, j\beta_d\, B_{zm}\, r^{1-m} + D_m r^{-m-1}$$
$$jE_{\phi d}(r,m) = (1/2)\,(m-1)^{-1}\, j\beta_d\, B_{zm}\, r^{1-m} + D_m\, r^{-m-1}$$

This was forcing 3 conditions on the two constants B_{zm} and D_m, an impossible situation.

This entire document is a total mess and useless due to my starting error, but I will keep it with that understand. Parts of it might be salvaged.

I have (I think) successfully computed the E and B fields inside a round wire, this is all written up in Appendix D sections D.1 through D.6. I wanted to add a short section describing the fields outside the wire just to finish things off. However, when I went to do this, I ended up with three boundary conditions to determine only two available constants, so I came into yet another problem. First, here is the method I used:

Ideas for an exterior solution writeup for Appendix B?

1. I think of the exterior solution as entirely driven by the interior solution and its current I. Unlike typical situations, you don't really have to consider both at once to get a solution to either mainly because β is much larger than β_d. [suspect this is not true]

2. I already know all the interior fields, and therefore I know them all on the r=a boundary. I know that both E and B satisfy $(\nabla^2 + \beta_d^2)\mathbf{E} = 0$ and $(\nabla^2 + \beta_d^2)\mathbf{B} = 0$ outside the wire. There is no gauge situation here, these are the E and B fields. [ok]

3. I can assume that the exterior fields have the same exp(***) wave dependence that the interior solutions do because they must match at the boundary. So then

$$\mathbf{E}(x,y,z,\omega) = e^{-j\beta_d z}\,\mathbf{E}(x,y,\omega)$$

for the exterior region. Therefore I end up with

$$(\nabla^2_{2D} + 0)\,\mathbf{E}(x,y,\omega) = 0. \qquad \text{ok}$$

4. This is still a *vector* ∇^2 operator so must be careful in cylindrical coordinates. What then have is basically the main line of Appendix D except we replace β^2 by β_d^2, and that changes the nature of ALL the component equations obtained with the vector ∇^2. If you look at the summary (D.1.20) the exterior solution situation will be this [I think all this is ok]

EXTERIOR

The Three Helmholtz Equations and the div E = 0 equation (in partial waves) (D.1.20)

$[\nabla^2\mathbf{E}]_z + \beta_d^2 E_z = 0$:

$$[r^2\partial_r^2 + r\,\partial_r - m^2]\,E_z(r,m) = 0 \qquad (D.1.15)_o$$

$[\nabla^2\mathbf{E}]_r + \beta_d^2 E_r = 0$:

$$[r^2\partial_r^2 + r\partial_r - (m^2+1)]\,E_r(r,m) - 2jm\,E_\varphi(r,m) = 0 \qquad (D.1.17)_o$$

$[\nabla^2\mathbf{E}]_\varphi + \beta_d^2 E_\varphi = 0$:

$$[r^2\partial_r^2 + r\partial_r - (m^2+1)]\,E_\varphi(r,m) + 2jmE_r(r,m) = 0 \qquad (D.1.18)_o$$

div $\mathbf{E} = 0$:

$$\partial_r\,[r\,E_r(r,m)] + jmE_\varphi(r,m) - j\beta_d\,r\,E_z(r,m) = 0 \qquad (D.1.19)$$

The last equation stays the same, but in the three previous we no longer even have Bessel equations! So I then have a completely new "solutions" section.

5. This is exactly the point I reached in "edit beta_d to k in App D.doc" Here is what I said there next:

The differential operators appearing in the above equations are no longer Bessel-style operators, they are Euler-style operators. Euler ODE's have the general form [$r^2\partial_r^2 + a\,r\,\partial_r + b$] f(r) = 0, and the solutions have this form from p 45 of Polyanin's excellent ODE compendium,

118. $x^2y''_{xx} + axy'_x + by = 0$.

The Euler equation.

Solution:

$$y = \begin{cases} |x|^{\frac{1-a}{2}}\left(C_1|x|^{\mu} + C_2|x|^{-\mu}\right) & \text{if } (1-a)^2 > 4b, \\ |x|^{\frac{1-a}{2}}\left(C_1 + C_2\ln|x|\right) & \text{if } (1-a)^2 = 4b, \\ |x|^{\frac{1-a}{2}}\left[C_1\sin(\mu\ln|x|) + C_2\cos(\mu\ln|x|)\right] & \text{if } (1-a)^2 < 4b, \end{cases}$$

where $\mu = \frac{1}{2}|(1-a)^2 - 4b|^{1/2}$.

For $E_z(r,m)$ the equation *(D.1.15)*$_o$ shown just above is in fact an Euler equation which has a = 1 and $b = -m^2$ so $\mu = m$ and the solution forms are these ($r \geq a$ outside the wire),

$E_z(r,m) = A_{zm}r^m + B_{zm}r^{-m}$ $\quad m > 0$

$E_z(r,0) = C_z\ln(r) + D_z$ $\quad m = 0$ $\quad$ ok

[I verified that this really is the solution with Maple] .

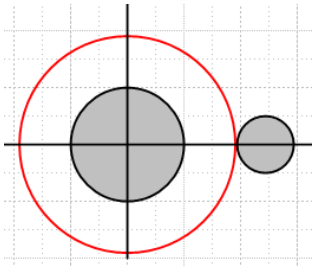
This E_z Helmholtz equation is the same as the radial equation one obtains when solving the 2D Laplace equation $\nabla^2 u = 0$ in separated polar coordinates, see for example Stakgold Vol II p 92 (6.7), and so what I see here are the familiar "atoms" of that problem.

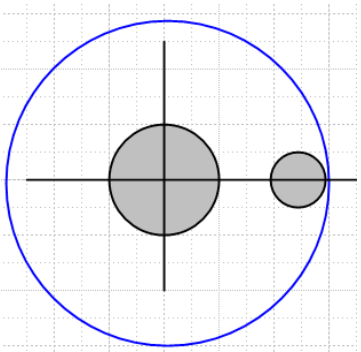
6. I have to then do some work to solve the other equations, same as in main Appendix D, since they are coupled. So I have to trace through how all that goes. For example go down to (D.2.5) for the E_r solution and just take it from there. I can get rid of the r^{+m} power certainly [wrong, see below] . So basically I just mimic the little sections of the main line. I will do that tomorrow.

Fri Dec 6, 2013

My physical picture is a coaxial cable with some very large radius call it R. Like an electric dipole at the origin surrounded by a large Great Sphere of radius R, I expect fields to drop off in r, not to be constant or to blow up, except maybe in the m = 0 case.

Note Added 12/12/13: The coaxial cable can only support the m = 0 wave, so it is the wrong "picture" for thinking about these boundary conditions. You need to use an off-center coax, or a twin lead or something like that. Here is a twin lead picture with the cylin coord system on the left conductor. This system will have $m \neq 0$ components, especially m = 1 I think:





You can talk about a Smythian Form within the red cylinder, but not outside because it hits the other transmission line conductor. So you could use atomic forms within the red cylinder. But this does not include r = ∞, so r = ∞ boundary conditions are not relevant! On the right you could use a Smythian Form outside the blue cylinder, and then the r = ∞ BC's *would* apply. But in this case you would have to think of that blue cylinder as the place where BC's apply, and this is not r = a! So either way, things don't work as described below.

Continuing the above with D.2 (a). There it says $E_z(r,m) = C_{zm} J_m(\beta' r)$, here we have $E_z(r,m) = B_{zm} r^{-m}$ at least for m > 0.

[But in the red cylinder, it would be the full $E_z(r,m) = A_{zm} r^m + B_{zm} r^{-m}$]

Then we go to (b)

$$[r^2 \partial_r^2 + r\partial_r - (m^2+1)]\, E_r(r,m) - 2jm\, E_\varphi(r,m) = 0$$

$$-jmE_\varphi(r,m) = [1 + r\partial_r]\, E_r(r,m) - r(j\beta_d)E_z(r,m)$$

It is the same as in the main doc but we have to set β' = 0. We get to

$$[r^2 \partial_r^2 + 3r\partial_r + (1-m^2)]\, E_r(r,m) = 2r(j\beta_d)E_z(r,m)\,. \qquad \text{(D.2.7)}$$

$$[r^2 \partial_r^2 + 3r\partial_r + (1-m^2)]\, E_r(r,m) = 2r(j\beta_d)\, B_{zm} r^{-m}\,. \qquad \text{(D.2.7)}$$

Then I have to solve this somehow. Restate

$$[r^2 \partial_r^2 + 3r\partial_r + (1-m^2)]\, E_r(r,m) = 2(j\beta_d)\, B_{zm} r^{1-m}\,. \qquad \text{(D.2.7)}$$

$$[r^2 \partial_r^2 + 3r\partial_r + (1-m^2)]\, E_r(r,m) = 2(j\beta_d)\, B_{zm} r^{1-m}\,. \qquad \text{(D.2.7)}$$

Maple gives the solution as (this must be particular plus homo) [exterior solution 1.mws]

$$E_r(r,m) = -(1/2)(m-1)^{-1} j\beta_d B_{zm}\, r^{1-m} + C_m r^{m-1} + D_m r^{-m-1}$$

and I verified that this does solve the ODE.

[But if we include $A_{zm}r^{m} + B_{zm}r^{-m}$ in $E_z(r,m)$ we then have

$$[r^2\partial_r^2 + 3r\partial_r + (1-m^2)]\, E_r(r,m) = 2(j\beta_d)\, A_{zm}r^{1+m} + 2(j\beta_d)\, B_{zm}r^{1-m} \qquad \text{(D.2.7)}$$

and this will have some other particular solution plus the same homo solution. I ran this in exterior solution 2. mws and here is the result

$$E1 := -\frac{1}{2}\frac{I\,b\,(r^{(1-m)}\,m + r^{(1-m)})\,B}{(1+m)(-1+m)} - 2\frac{I\,b\,Ar^{(1+m)}}{(1+m)(-1+m)} + _C1\,r^{(-1-m)} + _C2\,r^{(-1+m)}$$

where the first term is

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op(1,E1): simplify(%);
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$$-\frac{1}{2}\frac{Ib\,Br^{(1-m)}}{-1+m}$$

so then here is the solution

$$E_r(r,m) = -(j/2)(m-1)^{-1}\,\beta_d B_m\, r^{1-m} - 2j\,\beta_d(m^2-1)^{-1}A_m\, r^{1+m} + C_m\, r^{m-1} + D_m r^{-m-1}$$

where there are now four constants. This is the same as the original result except for the A_m term.

[What about m = 1? I tell Maple to solve $[r^2\partial_r^2 + 3r\partial_r]\, E_r(r,1) = 2(j\beta_d)\, B_{z1} r^0$ and it gives

$$E_r(r,m=1) = j\,\beta_d\, B_{zm}\,[\ln(r) - 1/2] + E + F/r^2$$

This seems to conflict with my dipole line charge result which gives $E_r = 1/r^3$.]

[What about m = 0 ? I tell Maple to solve $[r^2\partial_r^2 + 3r\partial_r + 1]\, E_r(r,0) = 2(j\beta_d)\, B_{z0} r$ and it gives

$$E_r(r,m=0) = (1/2)j\beta_d\, B_{z0}\, r + G/r + H\ln(r)/r$$

I guess I would then set $B_{z0} = 0$ and H = 0 and get $E_r(r,m=0) = G/r$ which is analogous to the line charge which has potential ln(r) and field 1/r.]

So it seems you might have to deal with m = 0 and m = ±1 separately.

Now I have already assumed m > 0. I will then assume that $C_m = 0$ since that term is constant or blows up [but that is wrong for our red cylinder, have to keep all terms] . Then we have

$$E_r(r,m) = -(1/2)(m-1)^{-1} j\beta_d\, B_{zm}\, r^{1-m} + D_m\, r^{-m-1}$$

$$E_r(r,m) = -(j/2)(m-1)^{-1}\,\beta_d B_m\, r^{1-m} - 2j\,\beta_d(m^2-1)^{-1}A_m\, r^{1+m} + C_m\, r^{m-1} + D_m r^{-m-1} \qquad \text{// new}$$

and this is then *like*

$$E_r(r,m) = a_m\, x^{-1}\, J_m(x) + \frac{K_m}{2}\, J_{m+1}(x)\ . \qquad \text{(D.2.15)}$$

where there are two unknown constants. I continue then to the E_φ solution

$$-jmE_\varphi(r,m) = [1 + r\partial_r]\, E_r(r,m) - r\,(j\beta_d)E_z(r,m)$$

$$= [1 + r\partial_r]\,\{-(1/2)(m-1)^{-1} j\beta_d\, B_{zm}\, r^{1-m} + D_m\, r^{-m-1}\} - r(j\beta_d)\, B_{zm}\, r^{-m}$$

$$= -(1/2)(m\text{-}1)^{-1} j\beta_d B_{zm} [1 + r\partial_r] r^{1-m} + D_m [1 + r\partial_r] r^{-m-1} - (j\beta_d) B_{zm} r^{-m+1}$$

$$-jmE_\varphi(r,m) = [1 + r\partial_r] E_r(r,m) - r (j\beta_d) E_z(r,m)$$

$$= [1 + r\partial_r] \{ -(j/2)(m\text{-}1)^{-1} \beta_d B_m r^{1-m} - 2j \beta_d (m^2\text{-}1)^{-1} A_m r^{1+m} + C_m r^{m-1} + D_m r^{-m-1} \}$$
$$- r (j\beta_d) \{ A_m r^m + B_m r^{-m} \}$$

$$= [1 + r\partial_r] \{ -(j/2)(m\text{-}1)^{-1} \beta_d B_m r^{1-m} - 2j \beta_d (m^2\text{-}1)^{-1} A_m r^{1+m} + C_m r^{m-1} + D_m r^{-m-1} \}$$
$$- (j\beta_d) \{ A_m r^{m+1} + B_m r^{-m+1} \}$$

Now

$$[1 + r\partial_r] r^{1-m} = r^{1-m} + r(1\text{-}m) r^{-m} = r^{1-m} + (1\text{-}m) r^{1-m} = (2\text{-}m) r^{1-m}$$

$$[1 + r\partial_r] r^{-m-1} = r^{-m-1} + r(-m\text{-}1) r^{-m-2} = r^{-m-1} + (-m\text{-}1) r^{-m-1} = -m r^{-m-1}$$

$$[1 + r\partial_r] r^s = r^s + r s r^{s-1} = (1+s) r^s$$

So then

$$-jmE_\varphi(r,m) = -(1/2)(m\text{-}1)^{-1} j\beta_d B_{zm} (2\text{-}m) r^{1-m} - D_m m r^{-m-1} - (j\beta_d) B_{zm} r^{-m+1}$$

$$= j\beta_d B_{zm} \{ -(1/2)(m\text{-}1)^{-1} (2\text{-}m) - 1 \} r^{1-m} - D_m m r^{-m-1}$$

$$= j\beta_d B_{zm} (1/2) \{ -(m\text{-}1)^{-1} (2\text{-}m) - 2 \} r^{1-m} - D_m m r^{-m-1}$$

$$= j\beta_d B_{zm} (1/2)(m\text{-}1)^{-1} \{ -(2\text{-}m) - 2(m\text{-}1) \} r^{1-m} - D_m m r^{-m-1}$$

$$= j\beta_d B_{zm} (1/2)(m\text{-}1)^{-1} \{ -2+m \ -2m+2 \} r^{1-m} - D_m m r^{-m-1}$$

$$= j\beta_d B_{zm} (1/2)(m\text{-}1)^{-1} \{ -m \} r^{1-m} - D_m m r^{-m-1}$$

$$-jmE_\varphi(r,m) =$$
$$\{ -(j/2)(m\text{-}1)^{-1} \beta_d B_m (2\text{-}m) r^{1-m} - 2j \beta_d (m^2\text{-}1)^{-1} A_m (2+m) r^{1+m} + C_m m r^{m-1} + D_m (-m) r^{-m-1} \}$$
$$- (j\beta_d) \{ A_m r^{m+1} + B_m r^{-m+1} \}$$

$$= -(j/2)(m\text{-}1)^{-1} \beta_d B_m (2\text{-}m) r^{1-m} - 2j \beta_d (m^2\text{-}1)^{-1} A_m (2+m) r^{1+m} + C_m m r^{m-1} + D_m (-m) r^{-m-1}$$
$$- (j\beta_d) A_m r^{m+1} - (j\beta_d) B_m r^{-m+1}$$

$$= r^{1-m} \{ -(j/2)(m\text{-}1)^{-1} (2\text{-}m) \beta_d B_m - (j\beta_d) B_m \}$$
$$+ r^{1+m} \{ -2j \beta_d (m^2\text{-}1)^{-1} A_m (2+m) - (j\beta_d) A_m \} + C_m m r^{m-1} + D_m (-m) r^{-m-1}$$

$$= r^{1-m} j \beta_d B_m (1/2) \{ -(m\text{-}1)^{-1} (2\text{-}m) - 2 \}$$
$$+ r^{1+m} j \beta_d A_m \{ -2 (m^2\text{-}1)^{-1} (2+m) - 1 \} + C_m m r^{m-1} + D_m (-m) r^{-m-1}$$

$$= r^{1-m}\, j\, \beta_d B_m (1/2)\, \{ \; -(2-m) - 2(m-1) \; \}/(m-1)$$
$$+ \; r^{1+m}\, j\, \beta_d A_m \, \{ -2 \;\; (2+m) - \; (m^2-1) \}/(m^2-1) \;\; + C_m\, m\, r^{m-1} \; + D_m(-m) r^{-m-1}$$

$$= r^{1-m}\, j\, \beta_d B_m (1/2)\, \{ \; -2+m-2m+2 \; \}/(m-1)$$
$$+ \; r^{1+m}\, j\, \beta_d A_m \, \{ -4 - 2m - \; m^2+1) \}/(m^2-1) \;\; + C_m\, m\, r^{m-1} \; + D_m(-m) r^{-m-1}$$

$$= r^{1-m}\, j\, \beta_d B_m (1/2)\, \{ -m \; \}/(m-1)$$
$$+ \; r^{1+m}\, j\, \beta_d A_m \, \{ -3 - 2m - \; m^2) \}/(m^2-1) \;\; + C_m\, m\, r^{m-1} \; + D_m(-m) r^{-m-1}$$

$$= - \; r^{1-m}\, j\, \beta_d B_m (1/2)\, \{ m \; \}/(m-1)$$
$$- \; r^{1+m}\, j\, \beta_d A_m \, \{ 3 + 2m + \; m^2 \}/(m^2-1) \;\; + C_m\, m\, r^{m-1} \; + D_m(-m) r^{-m-1}$$

and the above agrees with Maple. Then

$$-jmE_\varphi(r,m) = - \; r^{1-m}\, j\, \beta_d B_m (1/2)\, \{ m \; \}/(m-1)$$
$$- \; r^{1+m}\, j\, \beta_d A_m \, \{ 3 + 2m + \; m^2 \}/(m^2-1) \;\; + C_m\, m\, r^{m-1} \; + D_m(-m) r^{-m-1}$$

$$jmE_\varphi(r,m) = \; r^{1-m}\, j\, \beta_d B_m (1/2)\, \{ m \; \}/(m-1)$$
$$+ \; r^{1+m}\, j\, \beta_d A_m \, \{ 3 + 2m + \; m^2 \}/(m^2-1) \;\; - C_m\, m\, r^{m-1} \; + D_m \; m r^{-m-1}$$

$$jE_\varphi(r,m) = \; r^{1-m}\, j\, \beta_d B_m (1/2)\, \{ \; \}/(m-1)$$
$$+ \; r^{1+m}\, j\, \beta_d A_m \, \{ 3 + 2m + \; m^2 \}/[m(m^2-1)] \;\; - C_m \; r^{m-1} \; + D_m \; r^{-m-1}$$

$$jE_\varphi(r,m) = \; j\, \beta_d B_m (1/2) \frac{1}{m-1} \; r^{1-m} + \; j\, \beta_d A_m \frac{m^2+2m+3}{m(m^2-1)} \; r^{1+m} \;\; - C_m \; r^{m-1} \; + D_m \; r^{-m-1}$$

$$\frac{m^2+2m+3}{m(m^2-1)}$$

and then

$$-jE_\varphi(r,m) \; = - \, j\beta_d \, B_{zm} \; (1/2)(m-1)^{-1} \, r^{1-m} \; - \; D_m \, r^{-m-1}$$

$$= - \, (1/2) \, (m-1)^{-1} \; j\beta_d \, B_{zm} \;\; r^{1-m} \; - \; D_m \, r^{-m-1}$$

so that

$$jE_\varphi(r,m) = \; (1/2) \, (m-1)^{-1} \; j\beta_d \, B_{zm} \;\; r^{1-m} \; + \;\; D_m \, r^{-m-1}$$

which is a problem for m = 1, otherwise seems reasonable. All needs Maple checking!

[But now E_φ will have 4 constants and not two constants!]

So I end up with (for m >1)

$$E_{zd}(r,m) = B_{zm} r^{-m}$$

$$E_{rd}(r,m) = -(1/2)(m-1)^{-1}\ j\beta_d B_{zm}\ r^{1-m} + D_m r^{-m-1}$$

$$jE_{\varphi d}(r,m) = (1/2)(m-1)^{-1}\ j\beta_d B_{zm}\ r^{1-m} + D_m r^{-m-1}$$

Then we come to the boundary conditions. From the First Summary of the inside we know

First summary of the E field solutions **r = a** (D.2.21)

$$E_z(a,m) = -j(\beta'/\beta_d)\frac{K_m}{2} J_m(x_a) \qquad x = \beta' r \qquad \textit{(D.1.27)}$$

$$E_r(a,m) = a_m\ x_a^{-1} J_m(x_a) + \frac{K_m}{2} J_{m+1}(x_a)\ . \qquad \textit{(D.2.11)}$$

$$jE_\varphi(a,m) = -a_m\ x_a^{-1} J_m(x_a) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x_a)\ . \qquad \textit{(D.2.15)}$$

We then go look up the BC's to find:

$$E_z(a,m) = E_{zd}(a,m)$$

$$\xi E_r(a,m) = \xi_d E_{rd}(a,m) \qquad // = \xi (j\omega/\sigma) N_m \text{ according to (D.2.27)}$$

$$jE_\varphi(a,m) = jE_{\varphi d}(a,m) \qquad // = 0 \text{ according to (D.2.26)}$$

BUT, I assumed no current in the dielectric in order to get the first boundary condition at the wire surface in App D, so that means $\xi_d = \varepsilon_d$. And I know that inside the metal $\xi \approx (-j\sigma/\omega)$. So later I can insert these values, but keep the two ξ for the time being.

That then says

$$-j(\beta'/\beta_d)\frac{K_m}{2} J_m(x_a) = B_{zm} a^{-m}$$

$$\xi (j\omega/\sigma) N_m = \xi_d[-(1/2)(m-1)^{-1} j\beta_d B_{zm}\ a^{1-m} + D_m a^{-m-1}] \qquad \text{third equation}$$

$$0 = j\beta_d B_{zm}\ (1/2)(m-1)^{-1} a^{1-m} + D_m a^{-m-1} \qquad \text{second equation}$$

This seems a little strange because we then seem to have 3 conditions to determine B_{zm} and D_m .

[but in the red cylinder situation we have 3 conditions to determine 4 constants B_{zm}, A_{zm}, D_m and C_m.]

Let's go ahead and solve the first for B_{zm}

$$B_{zm} = - j\,(\beta'/\beta_d)\, a^m\, (j\omega/2\sigma)\, N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] J_m(x_a)$$

$$= (\beta'/\beta_d)\, a^m (\omega/2\sigma)\, N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] J_m(x_a)$$

Then we put this into the 2nd equation to get

$$- D_m\, a^{-m-1} = j\beta_d\, B_{zm}\ (1/2)(m-1)^{-1} a^{1-m}$$

$$= j\beta_d\, (\beta'/\beta_d)\, a^m (\omega/2\sigma)\, N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] J_m(x_a)\ (1/2)(m-1)^{-1} a^{1-m}$$

$$= j\beta'\, a^m\ (\omega/2\sigma)\ (1/2)(m-1)^{-1} a^{1-m}\, N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] J_m(x_a)$$

$$= j\beta'\, a\ (\omega/4\sigma)\ (m-1)^{-1} N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] J_m(x_a)$$

So then

$$D_m = - a^{m+2}\, j\beta'\, (\omega/4\sigma)\ (m-1)^{-1} N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] J_m(x_a)$$

What happens when we insert these into the third equation?

$$RHS = \xi_d[-(1/2)(m-1)^{-1} j\beta_d\, a^{1-m}\ (\beta'/\beta_d)\, a^m (\omega/2\sigma)\, N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] J_m(x_a)$$

$$- \xi_d\, a^{-m-1}\, a^{m+2}\, j\beta'\, (\omega/4\sigma)\ (m-1)^{-1} N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] J_m(x_a)$$

$$= - j\, \xi_d[(m-1)^{-1}\, a\beta'\, (\omega/4\sigma)\, N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] J_m(x_a)$$

$$- j\, \xi_d\, a\, \beta'\, (\omega/4\sigma)\ (m-1)^{-1} N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] J_m(x_a)$$

$$= -j\, \xi_d\, (\omega/4\sigma)\, N_m\ a\beta'\ (m-1)^{-1} \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] J_m(x_a)\ \{\ 1 + 1\ \}$$

$$= -j\, \xi_d\, (\omega/2\sigma)\, N_m\ a\beta'\ (m-1)^{-1} \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] J_m(x_a)$$

whereas

$$\text{LHS} = j\, 2\, \xi\, (\omega/2\sigma)\, N_m$$

so things don't seem to be working very well. But consider

$$\left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] J_m(x_a) = [\, J_{m-1}(x_a) - J_{m+1}(x_a)]\, J_m(x_a) / [J_{m+1}(x_a)\, J_{m-1}(x_a)]$$

$$= (2m/x_a)\, J_m(x_a)\, J_m(x_a) / [J_{m+1}(x_a)\, J_{m-1}(x_a)]$$

but this does not further simplify. So we have a problem.

Comments on the above method: I consider the exterior dielectric wave equation

$$(\, \nabla^2 + \beta_d^2\,)\, \mathbf{E}(r,\varphi,z,t) = 0, \qquad \text{(D.1.1)}$$

and as before I make this ansatz

$$\mathbf{E}(r,\varphi,z,t) = e^{j(\omega t - \beta_d z)}\, \mathbf{E}(r,\varphi)\ . \qquad \text{(D.1.2)}$$

and I make the same partial wave expansion,

$$\mathbf{E}(r,\varphi) = \sum_{m=-\infty}^{\infty} \mathbf{E}(r,m)\, e^{jm\varphi}\ .$$

This leads to the three Helmholtz equations in cylindricals plus the div E = 0 equation

$[\nabla^2 \mathbf{E}]_z + \beta_d^2\, E_z = 0$:

$$[r^2 \partial_r^2 + r\, \partial_r - m^2]\, E_z(r,m) = 0 \qquad \textit{(D.1.15)}$$

$[\nabla^2 \mathbf{E}]_r + \beta_d^2\, E_r = 0$:

$$[r^2 \partial_r^2 + r \partial_r - (m^2+1)]\, E_r(r,m) - 2jm\, E_\varphi(r,m) = 0 \qquad \textit{(D.1.17)}$$

$[\nabla^2 \mathbf{E}]_\varphi + \beta_d^2\, E_\varphi = 0$:

$$[r^2 \partial_r^2 + r \partial_r - (m^2+1)]\, E_\varphi(r,m) + 2jmE_r(r,m) = 0 \qquad \textit{(D.1.18)}$$

div $\mathbf{E} = 0$:

$$\partial_r\, [r\, E_r(r,m)] + jmE_\varphi(r,m) - j\beta_d\, r\, E_z(r,m) = 0 \qquad \textit{(D.1.19)}$$

These equations are the same as the inside equations but I have set $\beta' = 0$ since $\beta'^2 = \beta_{Helm}^2 - \beta_d^2$. But they are no longer Bessel equations. I solve these equations thinking for m > 1 and keeping only terms which don't blow up for large r and I get

$$E_{zd}(r,m) = B_{zm} r^{-m}$$

$E_{rd}(r,m) = -(1/2)(m-1)^{-1}\ j\beta_d\ B_{zm}\ r^{1-m} + D_m r^{-m-1}$
$jE_{\varphi d}(r,m) = (1/2)\ (m-1)^{-1}\ j\beta_d\ B_{zm}\ r^{1-m} + D_m\ r^{-m-1}$

[but for red cylinder, these are different and also have A_{zm} and C_m]

The first equation is homo and I keep only the decaying solution of this homo. To get the second equation, I use the same trick I used in App D: I look at the E_r equation and see jm $E_\varphi(r,m)$ sitting in it. I then replace this by a lin com of E_r and E_z from the div E = 0 equation. This then gives an equation which has only E_r and E_z, namely

$$[r^2\partial_r^2 + 3r\partial_r + (1-m^2)]\ E_r(r,m) = 2r\ (j\beta_d)E_z(r,m)\ .$$

I then install my solution for $E_z(r,m)$ into this to get a driven equation

$$[r^2\partial_r^2 + 3r\partial_r + (1-m^2)]\ E_r(r,m) = 2r\ (j\beta_d)\ B_{zm}r^{-m}\ .$$

I then solve this using Maple and after throwing out positive r powers the solution is

$$E_r(r,m) = -(1/2)(m-1)^{-1}j\beta_d\ B_{zm}\ r^{-m+1} + D_m r^{-m-1}$$

where only one of the two homo solution coefficients D_m appears. In this equation, the last term is the sort of m-pole expected power decay (m = 2 is octupole and is r^{-3} in 2D). The first term arises from the admixture of the E_z solution into the E_r solution caused by the Helmholtz equation and it has a lesser decay by two powers of r. When I tie this in, one can think of the E_z solution in the dielectric as being driven by the E_z in the wire just below the surface, and that is where this two-powers-less strange term is coming from. I then take the divE = 0 and it provides a solution for E_φ as follows,

$$jE_\varphi(r,m) = (1/2)\ (m-1)^{-1}\ j\beta_d\ B_{zm}\ r^{-m+1} + D_m\ r^{-m-1}$$

where the first term involves that same admixture of the E_z solution. So I then have the three solutions shown above which I repeat here:

$E_{zd}(r,m) = B_{zm}r^{-m}$
$E_{rd}(r,m) = -(1/2)(m-1)^{-1}\ j\beta_d\ B_{zm}\ r^{1-m} + D_m r^{-m-1}$
$jE_{\varphi d}(r,m) = (1/2)\ (m-1)^{-1}\ j\beta_d\ B_{zm}\ r^{1-m} + D_m\ r^{-m-1}$

My conundrum now arises because I am aware of three boundary conditions for these fields, but there are only two undetermined constants. [conundrum is resolved!] Thus boundary conditions are (really $\xi_d = \varepsilon_d$ since I assumed in my second inside BC that $\sigma_d = 0$, and also $\xi = (-j\sigma/\omega)$ inside the wire.

$E_z(a,m) = E_{zd}(a,m)$ // [tangential]
$\xi\ E_r(a,m) = \xi_d\ E_{rd}(a,m)$ // $= \xi\ (j\omega/\sigma)\ N_m$ according to (D.2.27)
$jE_\varphi(a,m) = jE_{\varphi d}(a,m)$ // = 0 according to (D.2.26) [tangential]

I then insert these facts:

(1) $E_{\varphi} = 0$ at the boundary, just one of my inside BC's.

(2) $E_r = (j\omega/\sigma)\, N_m$ at the boundary, another of my inside BC's

(3) insert $\xi = (-j\sigma/\omega)$ and $\xi_d = \varepsilon_d$

Doing this, E_r BC becomes

$$\xi\, E_r(a,m) = \xi_d\, E_{rd}(a,m)$$
$$(-j\sigma/\omega) E_r(a,m) = \varepsilon_d\, E_{rd}(a,m)$$
$$(-j\sigma/\omega)\,(j\omega/\sigma)\, N_m = \varepsilon_d\, E_{rd}(a,m)$$
$$N_m = \varepsilon_d\, E_{rd}(a,m)$$

so my three exterior BC's are then

boundary conditions:

$$E_{zd}(a,m) = -\,j\,(\beta'/\beta_d)\,\frac{K_m}{2}\,J_m(x) \qquad \text{where} \quad \frac{K_m}{2} = (j\omega/2\sigma)\, N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right]$$
$$E_{rd}(a,m) = N_m/\varepsilon_d$$
$$jE_{\varphi d}(a,m) = 0$$

solutions:

$$E_{zd}(r,m) = B_{zm} r^{-m}$$
$$E_{rd}(r,m) = -(1/2)(m-1)^{-1}\, j\beta_d\, B_{zm}\, r^{1-m} + D_m r^{-m-1}$$
$$jE_{\varphi d}(r,m) = (1/2)\,(m-1)^{-1}\, j\beta_d\, B_{zm}\, r^{1-m} + D_m\, r^{-m-1}$$

The obvious problem is that I have only two constants B_{zm} and D_m but I have to satisfy *three* BC's as shown. For example, consider that

$$E_{rd}(r,m) + jE_{\varphi d}(r,m) = 2\, D_m r^{-m-1}$$
$$E_{rd}(a,m) + jE_{\varphi d}(a,m) = 2\, D_m a^{-m-1}$$

Therefore

$$2\, D_m a^{-m-1} = N_m/\varepsilon_d \qquad \Rightarrow \qquad D_m = (1/2)\, a^{m+1}\, N_m/\varepsilon_d$$

Next, consider

$$E_{rd}(r,m) - jE_{\varphi d}(r,m) = -\,(m-1)^{-1}\, j\beta_d\, B_{zm}\, r^{1-m}$$
$$E_{rd}(a,m) - jE_{\varphi d}(a,m) = -\,(m-1)^{-1}\, j\beta_d\, B_{zm}\, a^{1-m}$$

Therefore

$$-\,(m-1)^{-1}\, j\beta_d\, B_{zm}\, a^{1-m} = N_m/\varepsilon_d \qquad \Rightarrow \qquad B_{zm} = j\,(m-1) a^{m-1} (1/\varepsilon_d\beta_d)\, N_m$$

The first solution must then be

$$E_{zd}(r,m) = j\,(m-1)a^{m-1}(1/\varepsilon_d\beta_d)\,N_m\,r^{-m}$$

dim chck: RHS = $L^{-1}\,L\,\varepsilon_0^{-1}\,C/m^2 = \varepsilon_0\,C/m^2 = m/F\,{*}Cm^{-2} = C\,F^{-1}m^{-1} = FV\,F^{-1}m^{-1} = V/m$

and then at r = a this becomes

$$E_{zd}(a,m) = j\,(m-1)(1/a)(1/\varepsilon_d\beta_d)\,N_m$$

But this is supposed to match

$$E_z(a,m) = -\,j\,(\beta'/\beta_d)\,\frac{K_m}{2}\,J_m(x_a) \qquad \text{where} \quad \frac{K_m}{2} = (j\omega/2\sigma)\,N_m\left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right]$$

which then requires

$$j\,(m-1)(1/a)(1/\varepsilon_d\beta_d)\,N_m = -\,j\,(\beta'/\beta_d)\,(j\omega/2\sigma)\,N_m\left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right]J_m(x)$$

or

$$(m-1)(1/a)(1/\varepsilon_d) = -\,\beta'\,(j\omega/2\sigma)\left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right]J_m(x_a)$$

For example, for m = 2 this requires that

$$(1/a)(1/\varepsilon_d) = -\,\beta'\,(j\omega/2\sigma)\left[\frac{1}{J_3(x_a)} - \frac{1}{J_1(x_a)}\right]J_2(x_a)$$

I can then set $\beta'^2 \approx \beta^2 = \omega^2\mu(-j\sigma/\omega) = -j\omega\mu\sigma$ to get

$$(1/a)(1/\varepsilon_d) = -\,\sqrt{-j\omega\mu\sigma}\;(j\omega/2\sigma)\left[\frac{1}{J_3(x_a)} - \frac{1}{J_1(x_a)}\right]J_2(x_a)$$

This condition is obviously not met!

Question: What is wrong in the above analysis? **12/6/13 1:20 PM**

I am now going to take a long hard look at Idea #1, it lasts several pages.

Idea #1. Suppose I were to assume that the inside solution had k in place of β_d where k was a to be determined parameter. How would that change the inside solution? These facts would still be true

$$E_r(r=a,m) = (j\omega/\sigma)\,N_m \qquad \text{where} \qquad N_m = \eta_m\,N_0 = \eta_m\,(k/2\pi\omega a)\,I \qquad \text{(D.2.26)}$$

$$E_\varphi(r=a,m) = 0 \qquad \text{(D.2.27)}$$

Anywhere β_d appears, it must be replaced by k. The wave is $e^{j(\omega t - kz)}$ now for inside and outside. Thus we have this inside z solution

$$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, \left[\frac{x_a J_m(x)}{J_{m+1}(x_a)} - \frac{x_a J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r \quad \beta'^2 = \beta^2 - k^2$$

Our boundary conditions on the exterior solutions would then be

$$E_{rd}(r=a,m) = (j\omega/\sigma)\,\eta_m\,(k/2\pi\omega a)\, I$$

$$E_{\phi d}(r=a,m) = 0$$

$$E_{zd}(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, \left[\frac{x_a J_m(x)}{J_{m+1}(x_a)} - \frac{x_a J_m(x)}{J_{m-1}(x_a)}\right]$$

We then have three conditions for what I suspect will be three constants: k, A_m, B_m .

If I now go through the processing of Appendix D where I make these changes

$$\beta' \rightarrow \qquad \beta'' = \beta_d^2 - k^2$$

$$r \rightarrow 0 \qquad r \rightarrow \infty \qquad \text{as BC}$$

I guess my E_z solution would have this form

$$E_z(r,m) = C_{zm}''\, H_m^{(1)}(\beta'' r)$$

since this would be the well-behaved Bessel function as $r \rightarrow \infty$. Then we would get

$$[r^2\partial_r^2 + 3r\partial_r + (1-m^2) + r^2\beta''^2)]\, E_{rd}(r,m) = K_m''\,\beta''\, r\, H_m^{(1)}(\beta'' r) \qquad \text{(D.2.8)}$$

$$K_m'' \equiv 2j\,(k/\beta'')\, C_{zm}''. \qquad \text{(D.2.9)}$$

$$E_{rd}(r,m) = x^{-1}\, f''_m(x) \qquad \text{(D.2.10)}$$

Then we end up with

$$[\, x^2\,\partial_x^2 + x\,\partial_x + (x^2 - m^2)]\, f''_m(x) = K''_m\, x''^2\, H_m^{(1)}(x'')$$

$$f''_m(x)^{particular} = (1/2)\, K_m\, [\, x''\, H_{m+1}^{(1)}(x'')]$$

$$f_m(x) = f_m(x)^{particular} + f_m(x)^{homogeneous} = (1/2)\, K_m''\, [\, x''\, H_{m+1}^{(1)}(x'')\,] + a''_m\, H_m^{(1)}(x'')]$$

$$E_{rd}(r,m) = a_m''\, x^{-1}\, H_m^{(1)}(x'') + \frac{K_m''}{2}\, H_{m+1}^{(1)}(x'')$$

The Bessel recursion is still true and this should lead us to

$$jE_{\varphi}(r,m) = - a_m'' \; x''^{-1} H_m^{(1)}(x'') + \left(\frac{K_m''}{2} + \frac{a_m''}{m}\right) H_{m+1}^{(1)}(x'')$$

So here are my new external solutions:

External

$$E_{zd}(r,m) = - j \, (\beta'''/k) \frac{K_m''}{2} \; H_m^{(1)}(x'') \qquad x'' = \beta'' r \qquad \beta''^2 = \beta_d^2 - k^2$$

$$E_{rd}(r,m) = a_m'' \; x''^{-1} H_m^{(1)}(x'') + \frac{K_m''}{2} \; H_{m+1}^{(1)}(x'')$$

$$jE_{\varphi d}(r,m) = - a_m'' \; x''^{-1} H_m^{(1)}(x'') + \left(\frac{K_m''}{2} + \frac{a_m''}{m}\right) H_{m+1}^{(1)}(x'')$$

Internal

$$E_z(r,m) = - j \, (\beta'/k) \frac{K_m}{2} \; J_m(x) \qquad x = \beta' r \qquad \beta'^2 = \beta^2 - k^2$$

$$E_r(r,m) = a_m \; x^{-1} J_m(x) + \frac{K_m}{2} \; J_{m+1}(x) \; . \qquad \textit{(D.2.11)}$$

$$jE_{\varphi}(r,m) = - a_m \; x^{-1} J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x) \qquad . \qquad \textit{(D.2.15)}$$

The boundary conditions are now

$E_{zd}(a,m) = E_z(a,m)$
$jE_{\varphi d}(a,m) = 0$
$\xi_d E_{rd}(a,m) = \xi \, E_r(a,m)$

Installing our forms we get

$$- j \, (\beta''/k) \frac{K_m''}{2} \; H_m^{(1)}(x''_a) = - j \, (\beta'/k) \frac{K_m}{2} \; J_m(x_a) \qquad // \; E_z \text{ condition}$$

$$- a_m'' \; x''^{-1}_a H_m^{(1)}(x'') + \left(\frac{K_m''}{2} + \frac{a_m''}{m}\right) H_{m+1}^{(1)}(x''_a) = 0 \qquad (2) \qquad // \; E_{\varphi} \text{ condition}$$

$$a_m'' \; x''^{-1}_a H_m^{(1)}(x''_a) + \frac{K_m''}{2} \; H_{m+1}^{(1)}(x''_a) = (\xi / \xi_d) (j\omega/\sigma) N_m \qquad (1) \qquad // \; E_r \text{ condition}$$

$x_a = \beta' a \qquad \beta'^2 = \beta^2 - k^2$
$x''_a = \beta'' a \qquad \beta''^2 = \beta_d^2 - k^2$

Notice that k cancels out in the first equation. We have three unknowns a_m'', K_m'' and k. We now mimic equations above (D.2.28) to find:

Add and subtract (1) and (2) to get

$$(2\frac{K_m''}{2}+\frac{a_m''}{m})\,H_{m+1}^{(1)}(x''_a) = (\xi/\xi_d)\,(j\omega/\sigma)\,N_m \qquad (3)$$

$$2\,a_m''\;x''^{-1}_a\,H_m^{(1)}(x''_a) - \frac{a_m''}{m}\,H_{m+1}^{(1)}(x''_a) = (\xi/\xi_d)\,(j\omega/\sigma)\,N_m \qquad (4)$$

Solve the second for a''_m :

$$a_m''\frac{1}{m}\;\{\;2m\,x''^{-1}_a\,H_m^{(1)}(x''_a) - H_{m+1}^{(1)}(x''_a)\} = (\xi/\xi_d)\,(j\omega/\sigma)\,N_m$$

$$a_m''\frac{1}{m}\;\{H_{m-1}^{(1)}(x''_a)\} = (\xi/\xi_d)\,(j\omega/\sigma)\,N_m$$

$$\frac{a_m''}{m} = (\xi/\xi_d)\,(j\omega/\sigma)\,N_m/\,H_{m-1}^{(1)}(x''_a) \qquad //\ (5)$$

$$(\frac{K_m''}{2}+\frac{a_m''}{m}) = (\xi/\xi_d)\,(j\omega/2\sigma)\,N_m\,[\frac{1}{H_{m+1}^{(1)}(x''_a)}+\frac{1}{H_{m-1}^{(1)}(x''_a)}]$$

$$\frac{K_m''}{2} = (\xi/\xi_d)\,(j\omega/2\sigma)\,N_m\,[\frac{1}{H_{m+1}^{(1)}(x''_a)}-\frac{1}{H_{m-1}^{(1)}(x''_a)}]$$

But now we have to satisfy the E_z condition

$$\beta''\frac{K_m''}{2}\;H_m^{(1)}(x''_a) = \beta'\frac{K_m}{2}\;J_m(x_a)$$

which is this

$$\beta''\,(\xi/\varepsilon_d)\;[\frac{1}{H_{m+1}^{(1)}(x''_a)}-\frac{1}{H_{m-1}^{(1)}(x''_a)}]\,H_m^{(1)}(x''_a)$$

$$= \beta'\,[\frac{1}{J_{m+1}(x_a)}-\frac{1}{J_{m-1}(x_a)}]\,J_m(x_a)$$

The above is the horrible equation which determines k which I guess could be complex.

$x_a = \beta' a \qquad \beta'^2 = \beta^2 - k^2$
$x''_a = \beta'' a \qquad \beta''^2 = \beta_d^2 - k^2$

$$\beta''\,(\xi/\varepsilon_d)\;\left[\frac{1}{H_{m+1}^{(1)}(x''_a)}-\frac{1}{H_{m-1}^{(1)}(x''_a)}\right]H_m^{(1)}(x''_a)$$

$$=\sqrt{\beta^2-k^2}\;\left[\frac{1}{J_{m+1}(x_a)}-\frac{1}{J_{m-1}(x_a)}\right]J_m(\sqrt{\beta^2-k^2}\,a)$$

But certainly $\beta >> k$ so we can rewrite the above as

$$\beta''\,(\xi/\varepsilon_d)\;\left[\frac{1}{H_{m+1}^{(1)}(x''_a)}-\frac{1}{H_{m-1}^{(1)}(x''_a)}\right]H_m^{(1)}(x''_a)$$

$$=\beta\left[\frac{1}{J_{m+1}(\beta a)}-\frac{1}{J_{m-1}(\beta a)}\right]J_m(\beta a)$$

and then the RHS is independent of k and seems to be a large number. If k is small relative to β_d as well, then perhaps we can use these limits

10.7.7

$$H_\nu^{(1)}(z)\sim -H_\nu^{(2)}(z)\sim -(i/\pi)\,\Gamma(\nu)(\tfrac{1}{2}z)^{-\nu},\quad \Re\nu>0.$$

Then in this quantity

$$\left[\frac{1}{H_{m+1}^{(1)}(x''_a)}-\frac{1}{H_{m-1}^{(1)}(x''_a)}\right]H_m^{(1)}(\sqrt{\beta_d^2-k^2}\,a)$$

the common factor of $-(i/\pi)$ cancels and we then have

$$=\left[\frac{1}{\Gamma(m+1)(x''_a/2)^{-(m+1)}}-\frac{1}{\Gamma(m-1)(x''_a/2)^{-(m-1)}}\right]\Gamma(m)(x_a/2)^{-m}$$

$$=\left[\frac{1}{\Gamma(m+1)(x''_a/2)^{-1}}-\frac{1}{\Gamma(m-1)(x''_a/2)}\right]\Gamma(m)$$

$$=\left[\frac{(x''_a/2)}{\Gamma(m+1)}-\frac{(x''_a/2)^{-1}}{\Gamma(m-1)}\right]\Gamma(m)$$

then our condition is

$$\beta''\,(\xi/\varepsilon_d)\;\left[\frac{(x''_a/2)}{\Gamma(m+1)}-\frac{(x''_a/2)^{-1}}{\Gamma(m-1)}\right]\Gamma(m)\;=\;\beta\left[\frac{1}{J_{m+1}(\beta a)}-\frac{1}{J_{m-1}(\beta a)}\right]J_m(\beta a)$$

or

$$(x''_a/a)\,(\xi/\varepsilon_d)\;\left[\frac{(x''_a/2)}{\Gamma(m+1)}-\frac{(x''_a/2)^{-1}}{\Gamma(m-1)}\right]\Gamma(m)\;=\;\beta\left[\frac{1}{J_{m+1}(\beta a)}-\frac{1}{J_{m-1}(\beta a)}\right]J_m(\beta a)$$

or

$$(2/a)\,(x''_a/2)\,(\xi/\varepsilon_d)\,\left[\frac{(x''_a/2)}{\Gamma(m+1)} - \frac{(x''_a/2)^{-1}}{\Gamma(m-1)}\right]\Gamma(m) = \beta\left[\frac{1}{J_{m+1}(\beta a)} - \frac{1}{J_{m-1}(\beta a)}\right]J_m(\beta a)$$

or

$$(2/a)\,(\xi/\varepsilon_d)\,\left[\frac{(x''_a/2)^2}{\Gamma(m+1)} - \frac{1}{\Gamma(m-1)}\right]\Gamma(m) = \beta\left[\frac{1}{J_{m+1}(\beta a)} - \frac{1}{J_{m-1}(\beta a)}\right]J_m(\beta a)$$

$$(2\xi/a\varepsilon_d)\,\left[\frac{(x''_a/2)^2}{\Gamma(m+1)} - \frac{1}{\Gamma(m-1)}\right]\Gamma(m) = \beta\left[\frac{1}{J_{m+1}(\beta a)} - \frac{1}{J_{m-1}(\beta a)}\right]J_m(\beta a)$$

$$\left[\frac{(x''_a/2)^2}{\Gamma(m+1)} - \frac{1}{\Gamma(m-1)}\right]\Gamma(m) = (a\varepsilon_d/2\xi)\,\beta\left[\frac{1}{J_{m+1}(\beta a)} - \frac{1}{J_{m-1}(\beta a)}\right]J_m(\beta a)$$

$$\frac{(x''_a/2)^2}{\Gamma(m+1)} = \frac{\Gamma(m)}{\Gamma(m-1)} + (a\varepsilon_d/2\xi)\,\beta\left[\frac{1}{J_{m+1}(\beta a)} - \frac{1}{J_{m-1}(\beta a)}\right]J_m(\beta a)$$

$$\frac{(x''_a/2)^2}{\Gamma(m+1)} = (m-1) + (a\varepsilon_d/2\xi)\,\beta\left[\frac{1}{J_{m+1}(\beta a)} - \frac{1}{J_{m-1}(\beta a)}\right]J_m(\beta a)$$

Now $\beta^2/\xi^2 \approx \omega^2\mu\,\xi/\xi^2 = \omega^2\mu/\xi \approx \omega^2\mu/(-j\sigma/\omega) = \omega^2\mu(-\omega/j\sigma) = j\,\omega^3\sigma$

$$\beta/\xi \approx j^{1/2}\omega^{3/2}\sigma^{1/2}$$

Well, the RHS does not seem particularly small, so this whole approximation with x''_a does not seem self consistent.

$$\sqrt{\beta_d^2 - k^2}\,(-j\sigma/\varepsilon_d\omega)\left[\frac{(x''_a/2)}{\Gamma(m+1)} - \frac{(x''_a/2)^{-1}}{\Gamma(m-1)}\right]\Gamma(m) = \beta\left[\frac{1}{J_{m+1}(\beta a)} - \frac{1}{J_{m-1}(\beta a)}\right]J_m(\beta a)$$

$x_a = \beta' a \qquad \beta'^2 = \beta^2 - k^2$
$x''_a = \beta'' a \qquad \beta''^2 = \beta_d^2 - k^2$

Now if β" is very small, maybe β"a is also small, so x"a is small, so then we have

$$\sqrt{\beta_d^2 - k^2}\,(-j\sigma/\varepsilon_d\omega)\left[\frac{(x''_a/2)^{-1}}{\Gamma(m-1)}\right]\Gamma(m) = \beta\left[\frac{1}{J_{m+1}(\beta a)} - \frac{1}{J_{m-1}(\beta a)}\right]J_m(\beta a)$$

$$\sqrt{\beta_d^2 - k^2}\,(-j\sigma/\varepsilon_d\omega)\left[(x''_a/2)^{-1}\right](m-1) = \beta\left[\frac{1}{J_{m+1}(\beta a)} - \frac{1}{J_{m-1}(\beta a)}\right]J_m(\beta a)$$

$$(x''_a/2) = \sqrt{\beta_d^2 - k^2}\,(-j\sigma/\varepsilon_d\omega)\,(m-1) \;/\; \left\{\beta\left[\frac{1}{J_{m+1}(\beta a)} - \frac{1}{J_{m-1}(\beta a)}\right]J_m(\beta a)\right\}$$

Now we have $\beta^2 = \omega^2\mu(-j\sigma/\omega) = -j\sigma\mu\omega$ so

$$(x''_a/2) = \sqrt{\beta_d^2 - k^2}\,(-j\sigma/\varepsilon_d\omega)\,(m\text{-}1) \;/\; \{\sqrt{-j\sigma\mu\omega}\;[\frac{1}{J_{m+1}(\beta a)} - \frac{1}{J_{m-1}(\beta a)}]\,J_m(\beta a)\}$$

But $\sqrt{\beta_d^2 - k^2} = \beta'' = x''$

OK, enough on this path without approximation.

Question: is there any reason to expect k to be close to β_d ?

People seem to assume this for a coaxial cable, I'll bet it is true.

Idea #2. Suppose I argue that although "large" by interior standards, E_z at the boundary is really quite small since inside it is E_z inside a good conductor. Then maybe set $B_{zm} = 0$ and take the resulting solutions as approximate:

We had:

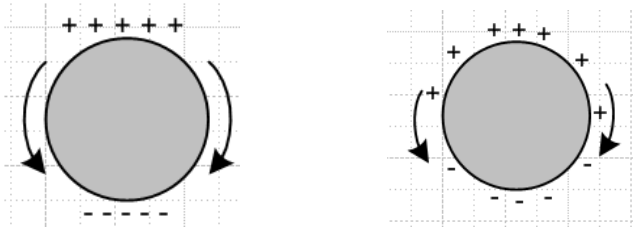
$$E_{zd}(r,m) = B_{zm} r^{-m}$$
$$E_{rd}(r,m) = -(1/2)(m\text{-}1)^{-1}\; j\beta_d\, B_{zm}\; r^{1-m} + D_m r^{-m-1}$$
$$jE_{\varphi d}(r,m) = (1/2)\,(m\text{-}1)^{-1}\; j\beta_d\, B_{zm}\, r^{1-m} + D_m\, r^{-m-1}$$

And I want to approximate this as

$$E_{zd}(r,m) \approx 0$$
$$E_{rd}(r,m) \approx D_m r^{-m-1}$$
$$jE_{\varphi d}(r,m) \approx D_m\, r^{-m-1}$$

The problem here is that the $E_{\varphi d}(a,m)$ BC then sets $D_m = 0$!!

Idea #3. Maybe my $E_\varphi(r,m) = 0$ is wrong! Think of an m = 1 moment.



At the ω rate, would not the plus charge flow as shown to the lower side, and would that not constitute some E_φ at the surface? There must be an E_φ field at the surface to make that surface charge move. I guess I was arguing that the charge was controlled by the fixed z pattern flowing down the wire. But

regardless of where the charge comes from or what is moving it, there must be some E_φ along that surface because E lines go from + charge to - charge.

I argued in my opening section of App D that $E_\varphi = 0$ because if not, $E_\varphi \neq 0$ would neutralize the charge. But think electrostatics where the right picture is a metal ball in an external E field. Even at DC you don't get neutralization because some OTHER E field is holding the charge there! And in our coaxial cable, there is an other field caused by the transmission line action.

Suppose then I were to forego this boundary condition altogether in my interior discussion. Then I just end up with these fields:

First summary of the E field solutions (D.2.21)

$$E_z(r,m) = -j(\beta'/\beta_d)\frac{K_m}{2} J_m(x) \qquad x = \beta' r \qquad \textit{(D.1.27)}$$

$$E_r(r,m) = a_m x^{-1} J_m(x) + \frac{K_m}{2} J_{m+1}(x) \; . \qquad \textit{(D.2.11)}$$

$$jE_\varphi(r,m) = -a_m x^{-1} J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x) \; . \qquad \textit{(D.2.15)}$$

Meanwhile, my external problem would with $k = \beta_d$ (simple case) would have these solutions,

$$E_{zd}(r,m) = B_{zm} r^{-m}$$
$$E_{rd}(r,m) = -(1/2)(m-1)^{-1} j\beta_d B_{zm} r^{1-m} + D_m r^{-m-1}$$
$$jE_{\varphi d}(r,m) = (1/2)(m-1)^{-1} j\beta_d B_{zm} r^{1-m} + D_m r^{-m-1}$$

Suppose I now impose these BC's :

$$E_{zd}(a,m) = E_z(a,m)$$
$$\xi_d E_{rd}(a,m) = \xi E_r(a,m)$$
$$jE_{\varphi d}(a,m) = jE_\varphi(a,m)$$

I still like the idea of tying the "free parameter" to N_m which says

$$E_r(r=a,m) = (j\omega/\sigma) N_m$$

and where the dielectric is assumed not to conduct to get this little rule as stated. So then I have these various equations:

(1) $B_{zm} a^{-m} = -j(\beta'/\beta_d)\frac{K_m}{2} J_m(x_a)$ // E_z match

(2) $\varepsilon_d\{-(1/2)(m-1)^{-1} j\beta_d B_{zm} r^{1-m} + D_m r^{-m-1}\} = \xi\{a_m x_a^{-1} J_m(x_a) + \frac{K_m}{2} J_{m+1}(x_a)\}$ // E_r

(3) $(1/2)\,(m\text{-}1)^{-1}\; j\beta_d\, B_{zm} a^{1-m} + D_m\, a^{-m-1} = -\,a_m\; x_a^{-1}\, J_m(x_a) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x_a)$ // jE_φ match

(4) $a_m\; x_a^{-1}\, J_m(x_a) + \frac{K_m}{2}\, J_{m+1}(x_a) = (j\omega/\sigma)\, N_m$ // tie-in to moment N_m

Now if I regard N_m as a given quantity, I have four equations for the four unknowns

a_m K_m B_{zm} D_m

Now let's solve these four equations. Rewrite as

(1) $B_{zm} = -\,j\, a^m(\beta'/\beta_d)\frac{K_m}{2}\, J_m(x_a)$ // E_z match

(2) $-\,(1/2)\,(m\text{-}1)^{-1}\; j\beta_d\, B_{zm}\, a^{1-m} + D_m\, a^{-m-1} = (\xi/\varepsilon_d)\,\{a_m\; x_a^{-1}\, J_m(x_a) + \frac{K_m}{2}\, J_{m+1}(x_a)\}$ // E_r

(3) $(1/2)\,(m\text{-}1)^{-1}\; j\beta_d\, B_{zm} a^{1-m} + D_m\, a^{-m-1} = -\,a_m\; x_a^{-1}\, J_m(x_a) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x_a)$ // jE_φ match

(4) $a_m\; x_a^{-1}\, J_m(x_a) + \frac{K_m}{2}\, J_{m+1}(x_a) = (j\omega/\sigma)\, N_m$ // tie-in to moment N_m

From (1) we can eliminate B_{zm} as shown.

Add (2)+(3) to get

$$2\, D_m\, a^{-m-1} = (\xi/\varepsilon_d)\,\{a_m\; x_a^{-1}\, J_m(x_a) + \frac{K_m}{2}\, J_{m+1}(x_a)\} + \{-\,a_m\; x_a^{-1}\, J_m(x_a) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x_a)\}$$

So if we can determine a_m and K_m then we know B_{zm} and D_m. Now compute (3) - (2) :

$$(m\text{-}1)^{-1}\; j\beta_d\, B_{zm} a^{1-m}$$
$$= -\,a_m\; x_a^{-1}\, J_m(x_a) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x_a) + (\xi/\varepsilon_d)\,\{-a_m\; x_a^{-1}\, J_m(x_a) - \frac{K_m}{2}\, J_{m+1}(x_a)\}$$

or

$$-(m\text{-}1)^{-1}\; j\beta_d\, j\, a^m(\beta'/\beta_d)\frac{K_m}{2}\, J_m(x_a)\; a^{1-m}$$
$$= -\,a_m\; x_a^{-1}\, J_m(x_a) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x_a) + (\xi/\varepsilon_d)\,\{-a_m\; x_a^{-1}\, J_m(x_a) - \frac{K_m}{2}\, J_{m+1}(x_a)\}$$

or

$$(m\text{-}1)^{-1}\; a\,\beta'\frac{K_m}{2}\, J_m(x_a)$$
$$= -\,a_m\; x_a^{-1}\, J_m(x_a) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x_a) + (\xi/\varepsilon_d)\,\{-a_m\; x_a^{-1}\, J_m(x_a) - \frac{K_m}{2}\, J_{m+1}(x_a)\}$$

or

$$\frac{K_m}{2}\ \{ (m\text{-}1)^{-1}\ a\beta'\ J_m(x_a) - J_{m+1}(x_a) + (\xi/\varepsilon_d)\ J_{m+1}(x_a) \}$$

$$= a_m \{ - x_a^{-1}\ J_m(x_a) + (1/m)\ J_{m+1}(x_a) - (\xi/\varepsilon_d)\ x_a^{-1}\ J_m(x_a) \}$$

or

$$\frac{K_m}{2}\ \{ (m\text{-}1)^{-1}\ a\beta'\ J_m(x_a) + [(\xi/\varepsilon_d)\text{-}1]\ J_{m+1}(x_a) \} = a_m \{ (1/m)\ J_{m+1}(x_a) - [(\xi/\varepsilon_d)+1]\ x_a^{-1}\ J_m(x_a) \}$$

which directly relates K_m and a_m. Now bring in the tie-in equation

$$a_m\ x_a^{-1}\ J_m(x_a) + \frac{K_m}{2}\ J_{m+1}(x_a) = (j\omega/\sigma)\ N_m$$

$$a_m \{ (1/m)\ J_{m+1}(x_a) - [(\xi/\varepsilon_d)+1]\ x_a^{-1}\ J_m(x_a) \} - \frac{K_m}{2}\ \{ (m\text{-}1)^{-1}\ a\beta'\ J_m(x_a)\ [(\xi/\varepsilon_d)\text{-}1] + J_{m+1}(x_a) \} = 0$$

We can finally solve for a_m and K_m , but it is extremely messy. Reduce clutter first

$$a_m\ x_a^{-1}\ J_m + \frac{K_m}{2}\ J_{m+1} = (j\omega/\sigma)\ N_m$$

$$a_m \{ (1/m)\ J_{m+1} - [\xi/\varepsilon_d+1]\ x_a^{-1}\ J_m \} - \frac{K_m}{2}\ \{ (m\text{-}1)^{-1}\ a\beta'\ J_m + [\xi/\varepsilon_d\text{-}1]\ J_{m+1} \} = 0$$

or

$$a_m\ x_a^{-1}\ J_m + \frac{K_m}{2}\ J_{m+1} = (j\omega/\sigma)\ N_m$$

$$a_m\ x_a^{-1}\ J_m \{ (1/m)\ x_a\ J_{m+1}/J_m - [\xi/\varepsilon_d+1] \} - \frac{K_m}{2}\ \{ (m\text{-}1)^{-1}\ a\beta'\ J_m + [\xi/\varepsilon_d\text{-}1]\ J_{m+1} \} = 0$$

Now replace $a_m\ x_a^{-1}\ J_m$ from the first in the second

$$\{ (j\omega/\sigma)\ N_m - \frac{K_m}{2}\ J_{m+1} \} \{ (1/m)\ x_a\ J_{m+1}/J_m - [\xi/\varepsilon_d+1] \} - \frac{K_m}{2}\ \{ (m\text{-}1)^{-1}\ a\beta'\ J_m + [\xi/\varepsilon_d\text{-}1]\ J_{m+1} \} = 0$$

$$\frac{K_m}{2} [\ J_{m+1} \{ (1/m)\ x_a\ J_{m+1}/J_m - [\xi/\varepsilon_d+1] \} + \{ (m\text{-}1)^{-1}\ a\beta'\ J_m + [\xi/\varepsilon_d\text{-}1]\ J_{m+1} \}]$$

$$= (j\omega/\sigma)\ N_m\ \{ (1/m)\ x_a\ J_{m+1}/J_m - [\xi/\varepsilon_d+1] \}$$

and finally we have a solution for $\frac{K_m}{2}$.

I am now ready for approximation! I think I will regard x_a and any J_n as of "nominal size". I think we can assume for sure that $\xi/\varepsilon_d+1 \approx \xi/\varepsilon_d$ since $\xi = -\ j\sigma/\omega$ and we always assume this is $>> \varepsilon_d$. So then

$$\frac{K_m}{2}\left[\ J_{m+1}\{\ (1/m)\ x_a\ J_{m+1}/J_m - \xi/\varepsilon_d\ \}\ +\ \{\ (m\text{-}1)^{-1}\ a\beta'\ J_m + [\xi/\varepsilon_d]\ J_{m+1}\}\ \right]$$

$$= (j\omega/\sigma)\ N_m\ \{\ (1/m)\ x_a\ J_{m+1}/J_m - [\xi/\varepsilon_d]\ \}$$

Next, we drop terms much smaller than the large number ξ/ε_d

$$\frac{K_m}{2}\left[\ J_{m+1}\{\ -\ [\xi/\varepsilon_d]\ \}\ +\ \{\ (m\text{-}1)^{-1}\ a\beta'\ J_m + [\xi/\varepsilon_d]\ J_{m+1}\}\ \right]\ = (j\omega/\sigma)\ N_m\ \{\ -\ [\xi/\varepsilon_d]\ \}$$

or

$$\frac{K_m}{2}\left[\ -\ J_{m+1}\ [\xi/\varepsilon_d]\ +(m\text{-}1)^{-1}\ a\beta'\ J_m + [\xi/\varepsilon_d]\ J_{m+1}\ \right]\ = (j\omega/\sigma)\ N_m\ \{\ -\ [\xi/\varepsilon_d]\ \}$$

or

$$\frac{K_m}{2}\left[\ (m\text{-}1)^{-1}\ a\beta'\ J_m\right]\ = -\ (j\omega/\sigma)\ N_m\ (\xi/\varepsilon_d)$$

or

$$\frac{K_m}{2}\ = -\ (m\text{-}1)(j\omega/\sigma)\ (1/a\beta')\ (\xi/\varepsilon_d)\ N_m/J_m$$

Now install

$$\beta'\ \approx\ \beta = e^{j3\pi/4}\sqrt{\omega\mu\sigma}$$

$$\xi \approx\ -\ j\sigma/\omega$$

Then

$$(j\omega/\sigma)\ (1/a\beta')\ (\xi/\varepsilon_d)\ = (j\omega/\sigma)(1/a\varepsilon_d)\ e^{-j3\pi/4}\ (1/\sqrt{\omega\mu\sigma}\ )\ (-\ j\sigma/\omega)$$

$$= (1/a\varepsilon_d)\ e^{-j3\pi/4}\ (1/\sqrt{\omega\mu\sigma}\ )\ \ = (1/\beta' a\varepsilon_d)$$

and then

$$\frac{K_m}{2}\ = -\ (m\text{-}1)\ (1/\beta' a\varepsilon_d)\ N_m/J_m \qquad\qquad N_m = \eta_m\ (\beta_d/2\pi\omega a)\ I$$

or

$$\frac{K_m}{2}\ = -\ (m\text{-}1)\ (1/\beta' a\varepsilon_d)\ \eta_m\ (\beta_d/2\pi\omega a)\ I\ /J_m$$

or

$$\frac{K_m}{2}\ = -\ (1/2\pi)(m\text{-}1)\ (\beta_d/\beta')(1/\omega a^2\varepsilon_d)\ \eta_m\ I\ /J_m$$

Meanwhile

$$B_{zm}\ a^{-m}\ = -\ j\ (\beta'/\beta_d)\ \frac{K_m}{2}\ J_m\ \ =\ -\ j\ (\beta'/\beta_d)\ J_m\ \frac{K_m}{2}$$

$= + j\,(\beta'/\beta_d)\,J_m\;(1/2\pi)(m-1)\,(\beta_d/\beta')(1/\omega a^2\varepsilon_d)\,\eta_m\;I/J_m$

$= + j\;(1/2\pi)(m-1)\,(1/\omega a^2\varepsilon_d)\,\eta_m\;I$

dim check: LHS = volt/m RHS = sec $m^{-2}(1/\varepsilon_0)$ amp

$=$ amp $\text{secm}^{-2}\text{m/f}$ $=$ amp secm^{-1} /f $=$ amp sec m^{-1} ohm sec^{-1} $=$ volt/m

Now want approximation for a_m . Recall

$$\frac{K_m}{2}\;\{\,(m-1)^{-1}\;a\beta'\,J_m(x_a) + [(\xi/\varepsilon_d)-1]\,J_{m+1}(x_a)\,\} \;= a_m\,\{\,(1/m)\,J_{m+1}(x_a) - [(\xi/\varepsilon_d)+1]\;x_a^{-1}\,J_m(x_a)\,\}$$

and now keep only large terms

$$\frac{K_m}{2}\;\{\,(m-1)^{-1}\;(\varepsilon_d/\xi)\,a\beta'\,J_m(x_a) + J_{m+1}(x_a)\,\} \;= -\,a_m\,x_a^{-1}\,J_m(x_a)$$

so

$$a_m\,x_a^{-1}\,J_m(x_a) \;= -\,\{\,(m-1)^{-1}\;(\varepsilon_d/\xi)\,a\beta'\,J_m(x_a) + J_{m+1}(x_a)\,\}\frac{K_m}{2}$$

$$a_m\,x_a^{-1} = -\,\{\,(m-1)^{-1}\;(\varepsilon_d/\xi)\,a\beta' + J_{m+1}(x_a)/\,J_m(x_a)\,\}\frac{K_m}{2}$$

$$a_m = -x_a\{\,(m-1)^{-1}(\varepsilon_d a\beta'/\xi) + J_{m+1}(x_a)/\,J_m(x_a)\,\}\frac{K_m}{2}$$

$$\frac{K_m}{2} \;= -\,(1/2\pi)(m-1)\,(\beta_d/\beta')(1/\omega a^2\varepsilon_d)\,\eta_m\;I/J_m$$

$$B_{zm}\;a^{-m} = j\;(1/2\pi)(m-1)\,(1/\omega a^2\varepsilon_d)\,\eta_m\;I$$

$$E_{zd}(r,m) = B_{zm}r^{-m} = j\;(1/2\pi)(m-1)\,(1/\omega a^2\varepsilon_d)\,\eta_m\;I\,(a/r)^m$$

Status at 7:45PM. As usual, my entire Appendix D has fallen apart again, just plain collapsed. I am once again unhappy with the $E_\varphi = 0$ condition on the surface, it seems wrong no. I liked it, then I didn't like it, then I liked it again, now I don't like it again. Poor stability here!

I can say this

$$E_z(r,m) = \;-\,j\,(\beta'/\beta_d)\,\frac{K_m}{2}\;J_m(x)$$

$$E_z(a,m) = -j\,(\beta'/\beta_d)\,\frac{K_m}{2}\,J_m(x_a)$$

$$\frac{E_z(r,m)}{E_z(a,m)} = \frac{J_m(\beta r)}{J_m(\beta a)} \qquad r \le a \qquad \beta r = e^{j3\pi/4}\,(\sqrt{2}/\delta)r = e^{j3\pi/4}\,z\,, \quad z = (\sqrt{2}/\delta)r$$

Does this show the skin effect? Yes it does, but you have to do the magnitude to see it

$$J_\nu(e^{j3\pi/4}z) = M_\nu(z)\,e^{j\theta_\nu(z)}\,. \tag{2.3.3}$$

$$J_\nu(\beta r) = M_\nu(\sqrt{2}\,r/\delta)\,e^{j\theta_\nu(z)}$$

So at least I can get some skin effect without evaluating K_m.

$$\left|\frac{E_z(r,m)}{E_z(a,m)}\right| = \left|\frac{J_m(\beta r)}{J_m(\beta a)}\right| = \frac{M_m(\sqrt{2}\,r/\delta)}{M_m(\sqrt{2}\,a/\delta)}$$

So you could plot this versus r to see about skin effect in higher modes. I would expect to see it still.