Integration check on (D.10.15) PhL 9.6.14

Here I verify that if I integrate $J_z = \sigma E_z$ with E_z in its large ω form below, I do in fact obtain the correct total current I in the wire! I have to assume that (a/δ) is large.

From (D.10.15),

$$E_z(r,\theta) = -(j\omega/\sigma)\sqrt{\frac{a}{r}}\, e^{(1+j)(r-a)/\delta}\; n(\theta)\,(\beta/k)$$

$$J_z(r,\theta) = -j\omega\sqrt{\frac{a}{r}}\, e^{(1+j)(r-a)/\delta}\; n(\theta)\,(\beta/k)$$

Let's do a little check on this. The integral of J_z should give I, but how is that going to work?

$$I = \int_{-\pi}^{\pi} d\theta \int_0^a r\,dr\, J_z(r,\theta) = -j\omega\,(\beta/k) \int_0^a r\,dr \sqrt{\frac{a}{r}}\, e^{(1+j)(r-a)/\delta} \int_{-\pi}^{\pi} d\theta\, n(\theta)$$

$$= -j\omega\,(\beta/k) * 2\pi N_0 \int_0^a r\,dr \sqrt{\frac{a}{r}}\, e^{(1+j)(r-a)/\delta}$$

Now perhaps let $x \equiv r/a$ so $dx = dr/a$ and $xdx = r/a\, dr/a = rdr/a^2$ and

$$(r-a)/\delta = (r/\delta) - (a/\delta) = (a/\delta)(r/a - 1) = (a/\delta)(x - 1)$$

The above is then

$$I = -j\omega\,(\beta/k) * 2\pi N_0 \int_0^1 a^2\, x\,dx/\sqrt{x} * e^{(1+j)(a/\delta)(x-1)}$$

$$= -j\omega\,(\beta/k) * 2\pi N_0\, a^2\, e^{-(1+j)(a/\delta)} \int_0^1 dx \sqrt{x}\; e^{(1+j)(a/\delta)x}$$

The integral is this, where $\alpha = (1+j)(a/\delta)$,

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Int(sqrt(x)*exp(alpha*x),x = 0..1);
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$$\int_0^1 \sqrt{x}\, e^{(\alpha x)}\, dx$$

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int(sqrt(x)*exp(alpha*x),x = 0..1);
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$$\frac{1}{2}\,\frac{2\, e^{\alpha}\sqrt{-\alpha} - \sqrt{\pi}\,\mathrm{erf}(\sqrt{-\alpha})}{\alpha\sqrt{-\alpha}}$$

which is the first time in lines doc I have ever seen the erf appear. If α is very large if $a/\delta >> 1$ then

$\mathrm{erf}(x) = 1 \qquad x \to \infty$ AS page 298

In this case the last blue expression above becomes

$$(1/2)\ [\ 2\ e^{\alpha}\sqrt{-\alpha}\ \ -\sqrt{\pi}\ ]\ /\ [\ \alpha\sqrt{-\alpha}\ ]$$

But again if α is large, the first term wins and we have

$$(1/2)\ [\ 2e^{\alpha}\sqrt{-\alpha}\ ]\ /\ [\ \alpha\sqrt{-\alpha}\ ]\ =\ [\ e^{\alpha}]\ /\ [\ \alpha]\ \ =e^{\alpha}/(\alpha)$$

$$= e^{(1+j)(a/\delta)}\ /\ [(1+j)(a/\delta)]$$

and we then find that

$$I = \ -\ j\omega\ (\beta/k)\ *\ 2\pi N_0\ a^2\ e^{-(1+j)(a/\delta)}\ *\ e^{(1+j)(a/\delta)}\ /\ [(1+j)(a/\delta)]$$

$$= -\ j\omega\ (\beta/k)\ *\ 2\pi N_0\ a^2\ /\ [(1+j)(a/\delta)]$$

Now I claim this above (D.10.15),

$$N_0 = (k/2\pi\omega a)\ I\ .\qquad\qquad (D.2.31)$$

so my check integral then says

$$I = \ -\ j\omega\ (\beta/k)\ *\ 2\pi(k/2\pi\omega a)\ I\ a^2\ /\ [(1+j)(a/\delta)]$$

$$=\ -\ j\omega\beta\ *\ 2\pi(1/2\pi\omega a)\ I\ a^2\ /\ [(1+j)(a/\delta)]$$

$$=\ -\ j\omega\beta\ *\ (1/\omega a)\ I\ a^2\ /\ [(1+j)(a/\delta)]$$

$$=\ -\ j\omega\beta\ *\ (1/\omega)\ I\ \ /\ [(1+j)(1/\delta)]$$

$$=\ -\ j\beta\ I\ \ /\ [(1+j)(1/\delta)]$$

But this requires that

$$1 = \ -\ j\beta/\ [(1+j)(1/\delta)]$$

$$(1+j)(1/\delta)\ \ = -j\beta$$

$$(1+j) = -j\delta\beta$$

But from above D.2.10 I know that

$$\delta\beta = (j-1)$$

so the check is then

$$(1+j) = -j(j-1) = 1+j$$

and I am duly amazed!