Old Section D.11 (d) PhL 10.6.14

Replaced by updates on 10.6.14, due to changes in D.9.

(d) Low frequency E fields

The fields in (D.2.33) quoted above then become,

$$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, [a\beta']\, f_m \qquad f_m = (r/a)^{|m|}\,(|m|+1)\,(2/\beta' a) \qquad f_0 = 4/(a\beta')$$
$$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (ak)\, g_m \qquad g_m = (r/a)^{|m|+1} + (r/a)^{|m|-1} \qquad g_0 = 2\,(r/a)$$
$$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (ak)\, h_m \qquad h_m = (r/a)^{|m|+1} - (r/a)^{|m|-1} \qquad h_0 = 0$$

or

$$E_z(r,m) = (1/2)\,\eta_m\, I\, R_{dc}\, (r/a)^{|m|}\,(|m|+1)$$
$$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (ak)\, [(r/a)^{|m|+1} + (r/a)^{|m|-1}] \qquad \text{(D.11.7)}$$
$$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (ak)\, [(r/a)^{|m|+1} - (r/a)^{|m|-1}]$$

$$E_z(r,0) = I\, R_{dc}$$
$$E_r(r,0) = (j/2)\, I\, R_{dc}\, (ak)\,(r/a)$$
$$E_\theta(r,0) = 0 \qquad \text{// low } \omega \text{ E fields}$$

If $n(\theta)$ is real and even in θ, we know from (D.10.4a) that,

$$E_z(r,\theta) = (1/4)\, I\, R_{dc}\, (a\beta')\, [\, f_0 + 2\,\Sigma_{m=1}^{\infty}\, f_m\, \eta_m\, \cos(m\theta)\,]$$
$$E_r(r,\theta) = (j/4)\, I\, R_{dc}\, (ak)\, [\, g_0 + 2\,\Sigma_{m=1}^{\infty}\, g_m\, \eta_m\, \cos(m\theta)\,]$$
$$E_\theta(r,\theta) = (1/4)\, I\, R_{dc}\, (ak)\, [\, h_0 + 2\,\Sigma_{m=1}^{\infty}\, h_m\, \eta_m\, \cos(m\theta)\,] \;. \qquad \textit{(D.10.4a)}$$

Inserting the expressions (D.11.6) then gives

$$E_z(r,\theta) = I\, R_{dc}\, \{\, 1 + \Sigma_{m=1}^{\infty}\, (r/a)^m\, (m+1)\, \eta_m\, \cos(m\theta)\,\}$$
$$E_r(r,\theta) = (j/2)\, I\, R_{dc}\, (ak)\, \{(r/a) + \Sigma_{m=1}^{\infty}\, [(r/a)^{m+1} + (r/a)^{m-1}]\, \eta_m\, \cos(m\theta)\,\}$$
$$E_\theta(r,\theta) = (1/2)\, I\, R_{dc}\, (ak)\, \{\, 0 \;+ \Sigma_{m=1}^{\infty}\, [(r/a)^{m+1} - (r/a)^{m-1}]\, \eta_m\, \cos(m\theta)\,\} \;. \qquad \text{(D.11.8)}$$

As for surface impedance, from (D.11.8) we find that, for low ω,

$$Z_s(\theta) \equiv E_z(a,\theta)/I = R_{dc}\, \{\, 1 + \Sigma_{m=1}^{\infty}\, (m+1)\, \eta_m\, \cos(m\theta)\,\} \qquad \text{(D.11.9)}$$

$$\langle Z_s(\theta)\rangle = R_{dc} \qquad \text{(D.11.10)}$$

and this last result certainly seems reasonable. Notice that, since $\mathbf{J} = \sigma\mathbf{E}$,

$$\frac{J_z(r,\theta)}{J_z(r,0)} = \frac{E_z(r,\theta)}{E_z(r,0)} = \frac{1 + \Sigma_{m=1}^{\infty}\, (r/a)^m\, (m+1)\, \eta_m\, \cos(m\theta)}{1 + \Sigma_{m=1}^{\infty}\, (r/a)^m\, (m+1)\, \eta_m} \;. \qquad \text{(D.11.11)}$$

Comment: The above expressions are for the case G = 0. To generalize to the case G>0, one makes the replacement $I \rightarrow I' = \sqrt{\xi_d/\varepsilon_d}\, I$ with an appropriate interpretation of k (see Section D.9 and (D.9.31)). This generalization has no effect on the ratio given in (D.11.11).

An anomaly. This last result is supposedly valid for very low ω. We would expect that in the limit ω→0 the above ratio should be exactly 1, at least for a non-conducting dielectric G = 0. This is so because we expect there to be no eddy currents at ω = 0 and these are the cause of J_z non-uniformity as discussed in Appendix P. We attribute our anomalous result (that J_z varies with θ) to the inaccuracy of the model at low ω, as outlined in section (a) above. It happens that in the limit ω→ 0 we also have I→ 0 when G = 0 (since then $Z_0 \rightarrow \infty$), but that is no justification for the anomalous result. See Chapter 7 for further discussion.

We ignore this anomaly and proceed with our task of finding the low frequency E fields in more detail. Recall from (4.11.18) that

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \qquad 1/Z_0 = \sqrt{\frac{G + j\omega C}{R + j\omega L}} \ . \qquad \textit{(4.12.18)}$$

For low ω we find from these expressions for $1/Z_0$ and from (D.11.1,2) that,

$$I = V/Z_0 \approx V\sqrt{G/R_{dc2}} \qquad k \approx -j\sqrt{R_{dc2}G} \qquad G > 0 \qquad \text{(D.11.12)}$$

$$I = V/Z_0 \approx V\sqrt{j\omega C/R_{dc2}} \qquad k \approx \sqrt{R_{dc2}C}\ \omega^{1/2}\, e^{-j\pi/4} \qquad G = 0 \qquad \text{(D.11.13)}$$

Inserting (D.11.12) into (D.11.7) gives, for G > 0,

Small ω limit of the E field solutions, G>0 : $R_{dc} = \dfrac{1}{\sigma\pi a^2}$ (D.11.14)

$$E_z(r,m) = (1/2)\ \eta_m V R_{dc}\sqrt{G/R_{dc2}}\ (r/a)^{|m|}\ (|m|+1) \qquad Z_0 = \sqrt{R_{dc2}/G}$$
$$E_r(r,m) = a\,(1/4)\,\eta_m V R_{dc} G\,[(r/a)^{|m|+1} + (r/a)^{|m|-1}]$$
$$E_\theta(r,m) = -ja\,(1/4)\,\eta_m V R_{dc}\ G\,[(r/a)^{|m|+1} - (r/a)^{|m|-1}] \qquad k = -j\sqrt{RG}$$
$$E_z(r,0) = V R_{dc}\sqrt{G/R_{dc2}}$$
$$E_r(r,0) = a\,(1/2)\,V R_{dc} G\,(r/a)$$
$$E_\theta(r,0) = 0$$

The fields are all finite and there are non-zero expressions for E_r and E_θ which account for the expected non-uniform flow of current into the dielectric through the conductor boundaries. We expect that $E_r(r,\theta)$ just outside the conductor boundary is a strong function of θ for closely spaced conductors (the capacitor problem), and thus so is $J_r(r,\theta)$. But $J_r(a,\theta)$ is continuous through the boundary at ω = 0, so we expect to see a strong dependence of $J_r(a,\theta)$ on θ inside the round wire, as indicated by $E_r(r,m)$ in (D.11.14).

Reader Exercise: Does (D.11.14) give the correct solution to the implied magnetostatics problem, or are there anomalies like the one noted above? Notice that Z_0 is certainly correct based on the reader exercise given in Appendix K (c). The "wave" decays in z according to $e^{-jkz} = \exp(-\sqrt{RG}\, z)$ which also seems reasonable.

Next, inserting (D.11.13) into (D.11.7) gives, for G = 0, this limiting form :

Small ω limit of the E field solutions, G=0 : $R_{dc} = \dfrac{1}{\sigma\pi a^2}$ (D.11.15)

$E_z(r,m) = (1/2)\,\eta_m\, V\, \omega^{1/2}\sqrt{jC/R}\; R_{dc}\, (r/a)^{|m|}\, (|m|+1)$
$E_r(r,m) = a\,(j/4)\,\eta_m\, V\, \omega\, C\; R_{dc}\; [(r/a)^{|m|+1} + (r/a)^{|m|-1}]$
$E_\theta(r,m) = a\,(1/4)\,\eta_m\, V\, \omega\, C\; R_{dc}\; [(r/a)^{|m|+1} - (r/a)^{|m|-1}]$

$E_z(r,0) = V\, \omega^{1/2}\sqrt{jC/R}\; R_{dc}$
$E_r(r,0) = a\,(j/2)\, V\, \omega\, C\; R_{dc}\, (r/a)$
$E_\theta(r,0) = 0$

As $\omega \to 0$, all fields vanish, corresponding to the fact that $Z_0 = \sqrt{R/j\omega C} \to \infty$ so $I = V/Z_0 \to 0$. In this situation the network model is just an infinite ladder of series resistors with no conductance cross pieces.

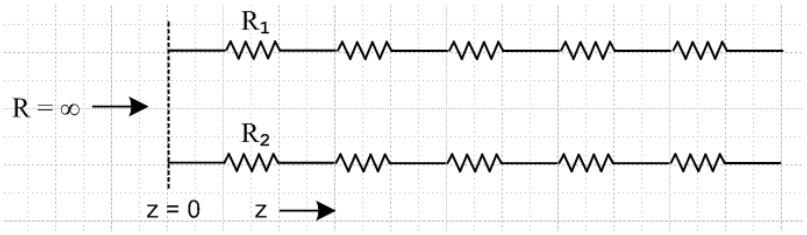

Fig D.8