Save low freq limit section of Appendix D PhL 12.4.13

Appendix D has grown a lot in the last month, and I think the following section should be removed. The limit itself has no particular interest to me, and nobody uses a transmission line in this limit anyway, so I will just park it here.

Other unused App D stuff is also stored here at the end.

D.5 The Low Frequency Limit **not very interesting, throw out!**

If we assume that ω is small enough that $|x_a| = |\beta'|a << 1$, we can greatly simplify the above results. For a good conductor, $\beta' \approx \beta = e^{j3\pi/4}(\sqrt{2}/\delta)$ so $|\beta'|a << 1 \Rightarrow (\sqrt{2}\, a/\delta) << 1$ or $(a/\delta) << 1$, which means the skin depth is much larger than the radius of the wire. In terms of frequency ω this condition is

$$\sqrt{\omega\mu\sigma}\ a << 1 \qquad \Rightarrow \qquad \omega << \frac{1}{\mu\sigma a^2}\ .$$

Using our ongoing example, for the Belden 8281 coax center conductor with a = .394 mm, $\mu = \mu_0 = 4\pi \times 10^{-7}$ henry/m, and $\sigma = 5.81 \times 10^7$ mho/m (copper), the limit requires that

$$\omega << \frac{10^6}{4\pi * 5.81 * (.394)^2} = 88 \text{ KHz}\ .$$

All limits come from the leading small-x term of $J_n(x)$ which is

$$J_m(x) \approx x^m/(2^m m!) \quad m \geq 0 \qquad \text{// Spiegel (24.2)} \qquad (D.5.1)$$

To avoid errors, we have Maple compute the limits of the various terms appearing in box ***.

```
A := xa*J(m,x)/J(m+1,xa): simplify(%);
```

$$\frac{xa\,J(m,x)}{J(m+1,xa)}$$

```
B := xa*J(m,x)/J(m-1,xa): simplify(%);
```

$$\frac{xa\,J(m,x)}{J(m-1,xa)}$$

```
C := (2*m/x)*J(m,x)/J(m-1,xa): simplify(%);
```

$$2\,\frac{m\,J(m,x)}{x\,J(m-1,xa)}$$

```
D_ := J(m+1,x)/J(m+1,xa): simplify(%);
```

$$\frac{J(m+1,x)}{J(m+1,xa)}$$

```
E := J(m+1,x)/J(m-1,xa): simplify(%);
```

$$\frac{J(m+1,x)}{J(m-1,xa)}$$

```
F := xa*J(0,x)/J(1,xa): simplify(%);
```

$$\frac{xa\,J(0,x)}{J(1,xa)}$$

```
G := J(1,x)/J(1,xa): simplify(%);
```

$$\frac{J(1,x)}{J(1,xa)}$$

```
J := (m,x) -> x^m/(2^m*m!);
```

$$J := (m,x) \to \frac{x^m}{2^m\,m!}$$

```
A := xa*J(m,x)/J(m+1,xa): simplify(%);
```

$$2\,xa^{(-m)}\,x^m\,(m+1)$$

```
B := xa*J(m,x)/J(m-1,xa): simplify(%);
```

$$\frac{1}{2}\,\frac{xa^{(2-m)}\,x^m}{m}$$

```
C := (2*m/x)*J(m,x)/J(m-1,xa): simplify(%);
```

$$x^{(m-1)}\,xa^{(-m+1)}$$

```
D_ := J(m+1,x)/J(m+1,xa): simplify(%);
```

$$x^{(m+1)}\,xa^{(-m-1)}$$

```
E := J(m+1,x)/J(m-1,xa): simplify(%);
```

$$\frac{1}{4}\,\frac{x^{(m+1)}\,xa^{(-m+1)}}{m\,(m+1)}$$

```
F := xa*J(0,x)/J(1,xa): simplify(%);
```

$$2$$

```
G := J(1,x)/J(1,xa): simplify(%);
```

$$\frac{x}{xa}$$

The left column is for visual check, the right column then provides the limits. Our limit is $x_a \ll 1$ and since $x \le x_a$ we also have $x \ll 1$, but the ratio $(x/x_a) = (r/a)$ is merely ≤ 1 and not $\ll 1$. From the column on the right above we read off the limits:

$$A = \frac{x_a J_m(x)}{J_{m+1}(x_a)} \approx 2(m+1)(x/x_a)^m$$

$$B = \frac{x_a J_m(x)}{J_{m-1}(x_a)} \approx (1/2m)(x/x_a)^m\, x_a^2 \approx 0 \qquad \text{since } x_a \ll 1$$

$$C = \frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} \approx (x/x_a)^{m-1}$$

$$D = \frac{J_{m+1}(x)}{J_{m+1}(x_a)} \approx (x/x_a)^{m+1}$$

$$E = \frac{J_{m+1}(x)}{J_{m-1}(x_a)} \approx \frac{1}{4m(m+1)}\,(x/x_a)^{m+1}\, x_a^2 \approx 0$$

$$F = \frac{x_a\ J_0(x)}{J_1(x_a)} \approx 2$$

$$G = \frac{J_1(x)}{J_1(x_a)} \approx (x/x_a) \qquad \text{(D.5.2)}$$

In terms of the letters A through G the E(r,m) fields from box *** are given by

$$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\,[A - B]$$
$$E_z(r,0) = (1/2)\, I\, R_{dc}\,[F]$$

$$E_r(r,m) = (j/4)\,(a\beta_d)\,\eta_m\, I\, R_{dc}\quad [\,C + D - E\,]$$
$$E_r(r,0) = (j/2)\,(a\beta_d)\, I\, R_{dc}\ [G]$$

$$E_\varphi(r,m) = (1/4)\,(a\beta_d)\,\eta_m\, I\, R_{dc}\ [\,-C + D + E\,]$$
$$E_\varphi(r,0) = 0 \qquad \text{(D.5.3)}$$

Installing the limits and setting $(x/x_a) = (r/a)$ we find

E Fields Inside a Round Wire (low frequency limit)

$$E_z(r,m) = (1/2)\,\eta_m\, I\, R_{dc}\ (m+1)(r/a)^m$$
$$E_z(r,0) = I\, R_{dc}$$

$$E_r(r,m) = (j/4)\,(a\beta_d)\,\eta_m\, I\, R_{dc}\quad [(r/a)^{m-1} + (r/a)^{m+1}\,]$$
$$E_r(r,0) = (j/2)\,(a\beta_d)\, I\, R_{dc}\ (r/a)$$

$$E_\varphi(r,m) = (1/4)\,(a\beta_d)\,\eta_m\, I\, R_{dc}\ [-(r/a)^{m-1} + (r/a)^{m+1}\,]$$
$$E_\varphi(r,0) = 0 \qquad \text{(D.5.4)}$$

The equation $E_z(r,0) = I\, R_{dc}$ states that in this low frequency limit the wire acts as a resistor having the expected resistance per unit length $R_{dc} = 1/(\sigma\pi a^2)$ as in (D.4.2). In the transmission line limit $a\beta_d << 1$ of interest to us, we see that there is always a small radial current which pumps the changing surface charge. In the m = 0 this radial current is linear in r, peaking at the surface.

In fact, if you integrate the radial current over the surface of a long piece of wire of length λ/ 2, taking into account the z dependence, you get 2I.

[I think the above has no basis in fact! At DC, we have ω = 0 and $E_r(a,0) = (j/2)\,(a\beta_d)\, I\, R_{dc}$. But $\beta_d = \omega\sqrt{\mu_d\xi_d} \approx \omega\sqrt{\mu_d\varepsilon_d} = 0$ at DC, so the integral of the radial current is 0. For m = 1, the radial current depends on η_1 which we don't know, and then $E_r(a,1) = (j/4)\,(a\beta_d)\,\eta_m\, I\, R_{dc}(2)$ and we have no conclusion at all. Yes it is *proportional* to I. I don't know where this strange claim came from! Throwing in the sinφ angle and integrating over half a wavelength is not going to rescue this weird claim.]

See Section 3.7, Figure 2 for a drawing of the current we have just computed.

continue here after reading Chapter 3.

********** other discarded stuff from Appendix D **********

The Maxwell curl **E** equation is not tested since we know it is valid since it was used to compute B.

Inserting these coefficients we find

$$[(j\omega/2\sigma)\, N_m\,]^{-1}(\omega/\beta')B_z(r,m) \;=\; [\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\,]\, J_m(x)$$

$$[(j\omega/2\sigma)\, N_m\,]^{-1}\,(\omega/j\beta')B_r(r,m) = (\, m[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\,]\,\frac{\beta'}{\beta_d} - \frac{\beta_d}{\beta'}\, 2m\, \frac{1}{J_{m-1}(x_a)})\, J_m(x)/x + \frac{\beta_d}{\beta'}\,[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\,]\, J_{m+1}(x)$$

$$[(j\omega/2\sigma)\, N_m\,]^{-1}\,(\omega/\beta')B_\varphi(r,m) = -\,(\, m[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\,]\,\frac{\beta'}{\beta_d} - \frac{\beta_d}{\beta'}\, 2m\, \frac{1}{J_{m-1}(x_a)})\, J_m(x)/x + [\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\,]\,(\frac{\beta_d}{\beta'} + \frac{\beta'}{\beta_d})\, J_{m+1}(x)$$

Now use (D.2.28) on the left and factor out m in two places to get

$$[(j/4)\,(a\beta_d)\,\eta_m\, I\, R_{dc}]^{-1}(\omega/\beta')B_z(r,m) \;=\; [\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\,]\, J_m(x)$$

$$[(j/4)\,(a\beta_d)\,\eta_m\, I\, R_{dc}\,]^{-1}\,(\omega/j\beta')B_r(r,m) = m\,\{\;[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\,]\,\frac{\beta'}{\beta_d} - \frac{\beta_d}{\beta'}\, 2\,\frac{1}{J_{m-1}(x_a)}\,\}\, J_m(x)/x + \frac{\beta_d}{\beta'}\,[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\,]\, J_{m+1}(x)$$

$$[(j/4)\,(a\beta_d)\,\eta_m\, I\, R_{dc}\,]^{-1}\,(\omega/\beta')B_\varphi(r,m) = -\,m\,\{\,[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\,]\,\frac{\beta'}{\beta_d} - \frac{\beta_d}{\beta'}\, 2\,\frac{1}{J_{m-1}(x_a)}\,\}\, J_m(x)/x + [\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\,]\,(\frac{\beta_d}{\beta'} + \frac{\beta'}{\beta_d})\, J_{m+1}(x)$$

Now move all factors to the right side

$$B_z(r,m) = (\beta'/\omega)(j/4)\,(a\beta_d)\,\eta_m\, I\, R_{dc}\;[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\,]\, J_m(x)$$

$$B_r(r,m) \;= (j\beta'/\omega)(j/4)\,(a\beta_d)\,\eta_m\, I\, R_{dc}\,* \; m\,\{\;[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\,]\,\frac{\beta'}{\beta_d} - \frac{\beta_d}{\beta'}\, 2\,\frac{1}{J_{m-1}(x_a)}\,\}\, J_m(x)/x + \frac{\beta_d}{\beta'}\,[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\,]\, J_{m+1}(x)$$

$$B_\varphi(r,m) \;= (\beta'/\omega)(j/4)\,(a\beta_d)\,\eta_m\, I\, R_{dc}\,* \; -\,m\,\{\,[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\,]\,\frac{\beta'}{\beta_d} - \frac{\beta_d}{\beta'}\, 2\,\frac{1}{J_{m-1}(x_a)}\,\}\, J_m(x)/x + [\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\,]\,(\frac{\beta_d}{\beta'} + \frac{\beta'}{\beta_d})\, J_{m+1}(x)$$

We can replace

$$(j\omega/2\sigma)\, N_m = (j/4)\,(a\beta_d)\,\eta_m\, I\, R_{dc} \qquad (D.2.28)$$

Therefore we get

$$(\omega/\beta')B_z(r,m) = (j\omega/2\sigma)\, N_m \left[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\right] J_m(x)$$

$$(\omega/j\beta')B_r(r,m) = + \left(m(j\omega/2\sigma)\, N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] \frac{\beta'}{\beta_d} - \frac{\beta_d}{\beta'}\, (j\omega/2\sigma)\, 2m\, N_m \frac{1}{J_{m-1}(x_a)}\right) J_m(x)/x + \frac{\beta_d}{\beta'}\, (j\omega/2\sigma)\, N_m \left[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\right]$$

$$(\omega/\beta')B_\varphi(r,m) = - \left(m(j\omega/2\sigma)\, N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] \frac{\beta'}{\beta_d} - \frac{\beta_d}{\beta'}\, (j\omega/2\sigma)\, 2m\, N_m \frac{1}{J_{m-1}(x_a)}\right) J_m(x)/x + (j\omega/2\sigma)\, N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] \left(\frac{\beta_d}{\beta'} + \frac{\beta'}{\beta_d}\right) J_{m+1}(x)$$

Replacing

or

$$(1/\beta')B_z(r,m) = (j/2\sigma)\, N_m \left[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\right] J_m(x)$$

$$(\omega/j\beta')B_r(r,m) = + \left(m(j\omega/2\sigma)\, N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] \frac{\beta'}{\beta_d} - \frac{\beta_d}{\beta'}\, (j\omega/2\sigma)\, 2m\, N_m \frac{1}{J_{m-1}(x_a)}\right) J_m(x)/x + \frac{\beta_d}{\beta'}\, (j\omega/2\sigma)\, N_m \left[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\right]$$

$$(\omega/\beta')B_\varphi(r,m) = - \left(m(j\omega/2\sigma)\, N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] \frac{\beta'}{\beta_d} - \frac{\beta_d}{\beta'}\, (j\omega/2\sigma)\, 2m\, N_m \frac{1}{J_{m-1}(x_a)}\right) J_m(x)/x + (j\omega/2\sigma)\, N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] \left(\frac{\beta_d}{\beta'} + \frac{\beta'}{\beta_d}\right) J_{m+1}(x)$$

$J_{m+1}(x)$

so then

$$(\omega/j\beta_d)B_r(r,m) = \left(m\frac{K_m}{2}\frac{\beta'}{\beta_d} - \frac{\beta_d}{\beta'} a_m \right) J_m(x)/x + (\beta_d/\beta') \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x)$$

Finally for B_φ,

$$(\omega/\beta_d)B_\varphi(r,m) = \left(m\frac{K_m}{2}\frac{1}{r\beta_d} - \frac{\beta_d}{r\beta'^2} a_m \right) J_m(x)$$

$$= \left(m\frac{K_m}{2}\frac{\beta'}{\beta_d} - a_m \right) J_m(x)/x + (\beta_d/\beta') \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x)$$

$$= \left(m\frac{K_m}{2}\frac{\beta'}{r\beta'\beta_d} - \frac{\beta_d}{r\beta'^2} a_m \right) J_m(x) + (\beta_d/\beta') \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x)$$

$$(\omega/\beta_d)B_\theta(r,m) = -\left[(\beta'^2/\beta_d^2) \; m\frac{K_m}{2} - a_m \right] J_m(x)/x + \frac{K_m}{2}(1 + \beta'^2/\beta_d^2) J_{m+1}(x)$$

$$jE_\varphi(r,m) = (j/4)(a\beta_d)\,\eta_m\, I\, R_{dc} \left\{ - \frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m+1}(x)}{J_{m-1}(x_a)} \right] \right\}$$

We summarize these results in a box. The m=0 results are all obtainable from the general expressions with m = 0 using $J_{-1} = -J_1$:

E Fields Inside a Round Wire

$x = \beta' r \qquad \beta'^2 = \beta^2 - \beta_d^2 \approx \beta^2 \qquad \beta = \omega\sqrt{\mu\xi} \qquad \xi = [\varepsilon + \sigma/(j\omega)] \approx \sigma/(j\omega)$ conductor

$x_a = \beta' a \qquad \beta_d = \omega\sqrt{\mu_d\xi_d} \qquad \xi_d = [\varepsilon_d + \sigma_d/(j\omega)] \approx \varepsilon_d$ dielectric

$\mathbf{E}(r,\varphi,z,t) = e^{j(\omega t - \beta_d z)}\,\mathbf{E}(r,\varphi)$ *(D.1.2)* $\qquad \mathbf{E} = E_r\hat{\mathbf{r}} + E_\varphi\hat{\boldsymbol{\varphi}} + E_z\hat{\mathbf{z}}$ (for any arguments)

$$\mathbf{E}(r,\varphi) = \mathbf{E}(r,0) + 2\sum_{m=1}^{\infty}[\mathrm{Re}\{\mathbf{E}(r,m)\}\cos(m\varphi) - \mathrm{Im}\{\mathbf{E}(r,m)\}\sin(m\varphi)] = \text{real} \qquad (D.1.4)$$

where:

$$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\left[\frac{x_a J_m(x)}{J_{m+1}(x_a)} - \frac{x_a J_m(x)}{J_{m-1}(x_a)}\right] \qquad a = \text{radius} \qquad \eta_m \equiv \frac{N_m}{N_0}$$

$$E_z(r,0) = (1/2)\, I\, R_{dc}\left[\frac{x_a\, J_0(x)}{J_1(x_a)}\right] \qquad // = (2.2.22) \text{ for } \beta_d = 0 \qquad \eta_0 = 1$$

$$E_r(r,m) = (j/4)\,(a\beta_d)\,\eta_m\, I\, R_{dc}\left[\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m+1}(x)}{J_{m-1}(x_a)}\right]$$

$$E_r(r,0) = (j/2)\,(a\beta_d)\, I\, R_{dc}\left[\frac{J_1(x)}{J_1(x_a)}\right] \qquad R_{dc} = \frac{1}{\sigma\pi a^2}$$

$$E_\varphi(r,m) = (1/4)\,(a\beta_d)\,\eta_m\, I\, R_{dc}\left[-\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m+1}(x)}{J_{m-1}(x_a)}\right]$$

$$E_\varphi(r,0) = 0$$

(D.4.6)