This material is more or less now installed as lines doc Section D.9.

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D.9 About the boundary condition $E_r(a,\theta) = (j\omega/\sigma)\, n(\theta)$.

The "charge pumping boundary condition" appears in (D.2.23) and here we want to examine it more closely. Our concern is that the derivation of (D.2.23) ignores surface currents that we know exist on the surface of a transmission line conductor as the surface charge moves around in response to tangential E fields. The first issue then is to define and quantify the nature of these surface currents.

(a) The notion of Debye Surface Currents

We continue in the context of our classical treatment of the conductor surface. In Appendix E it is pointed out that the surface charge on a transmission line conductor exists in an incredibly thin surface layer we shall call the Debye layer for want of a better name. For copper the thickness λ_D of this layer is on the order of one atomic radius. In addition to the normal conduction electrons, this thin layer contains extra free electrons that are piled up just below the surface (negative surface charge) or are depleted from this thin region (positive surface charge), as shown by the red curve in Fig E.1. We want first so show :

Fact 1: In a good conductor, the volume density of free electron carriers piled up at a surface (to make up the surface charge) is negligible compared to the volume density of conduction electrons. (D.9.1)

Proof: From (E.7) the free charge density in the Debye layer (assume x is the inward surface normal direction) is given by $\rho(x) = \rho(0)\, e^{-x/\lambda_D}$. The effective free surface charge n is then given by

$$n = \int_0^\infty dx\, \rho(x) = \rho(0) \int_0^\infty dx\, e^{-x/\lambda_D} = \rho(0)\lambda_D .$$

The free electron density n_e is then

$$n_e = \rho(0)/e = n / (e\lambda_D) .$$

As a typical example, consider a parallel plate capacitor with close plate spacing s. The E field in the gap is E = V/s and the surface charge density from (1.1.47) is $n = \varepsilon E = \varepsilon V/s$. For V = 10 volts and s = 1 mm we find

$$n = \varepsilon_0 V/s = 8.85 \times 10^{-12} * 10 / 10^{-3} \approx 10^{1-12+1+3} = 10^{-7}\ \text{Coul/m}^2 .$$

Then the free electron density is

$$n_e = n / (e\lambda_D) \approx 10^{-7}\ \text{Cou/m}^2 / [\ 1.6 \times 10^{-19}\ \text{Coul} * 10^{-10}\text{m}]$$

$$\approx 0.6 * 10^{-7+19+10} \approx 10^{22}\ \text{electrons/m}^3$$

As noted in (N.1.2), in copper the conduction electron density (one electron per atom) is 10^{29} /m^3, QED.

Corollary: The conductivity σ_D inside the Debye later is basically the same as σ outside that layer.

(D.9.2)

Proof: From (N.1.10) conductivity is $\sigma = (nq^2\tau/m)$ where n is the electron density. The Fact above shows that this density is the same in the Debye layer as in the bulk conductor, so $\sigma_D = \sigma$. (We ignore the possibility that the collision time τ could differ in the Debye layer vs. in the bulk volume.) QED

Consider now this drawing which shows a tiny slice of width dx of a piece of a transmission line conductor cross section at its surface. The yellow Debye surface charge layer is greatly exaggerated in thickness. Recall from the comment below Fig E.2 that at 100 GHz one has $\delta \approx 4000\ \lambda_D$ so $\delta >> \lambda_D$ at all frequencies of transmission line interest.

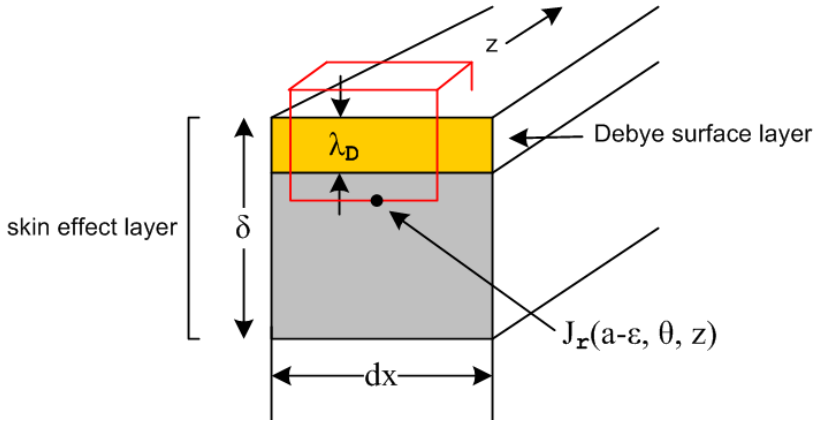

Fig D.6

The Debye layer holds the surface charge, and when this surface charge moves, one has a Debye surface current. We now show :

Fact 2: The total current in the Debye layer is negligible compared to that in the skin effect layer.

(D.9.3)

Proof: The field E_z is transverse to the conductor surface, so we know from (1.1.41) that it is continuous through the (gradual) boundary at the bottom of the Debye layer. Then the ratio of the currents in the two layers is,

$$\frac{I^D}{I^\delta} = \frac{J_z^D\,\lambda_d\,dx}{J_z^\delta\,\delta\,dx} = \frac{\sigma_D E_z^D\,\lambda_d}{\sigma E_z^\delta\,\delta} = \frac{\sigma_D}{\sigma}\,\frac{E_z}{E_z}\,\frac{\lambda_D}{\delta} \approx 1 * 1 * \frac{\lambda_D}{\delta} = \frac{\lambda_D}{\delta} << 1\ .$$ QED

(b) The role of Debye Surface Currents in the Charge Pumping Boundary Condition

Now referring to the Debye surface currents as K_z^D and K_θ^D we reconsider the derivation of the charge pumping boundary condition of (D.2.24) where we had this figure

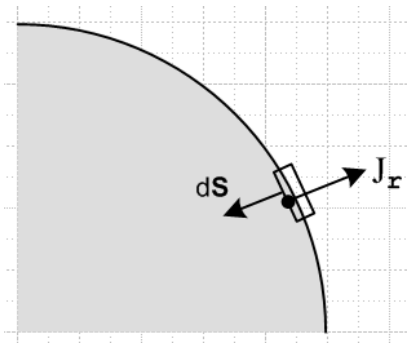

Fig D.2

If we include the Debye surface currents in the θ and z direction in our application of continuity,

$$\text{div } \mathbf{J} = -j\omega\rho \quad \Leftrightarrow \quad -j\omega[\int_V \rho\, dV] = \int_S \mathbf{J} \bullet d\mathbf{S} \qquad (D.2.22)$$

the result is

$$-j\omega\, n(\theta,z) = -J_r(r=a-\varepsilon, \theta, z) + \partial_z K_z(\theta,z) + a\,\partial_\theta K_\theta(\theta,z)\,. \qquad (D.9.4)$$

where we assume that the dielectric outside the round wire is vacuum with $\sigma_d = 0$. The gaussian box selected here is that shown in red in Fig ***. The bottom face lies below the Debye layer so $J_r(a-\varepsilon, \theta, z)$ is the value of J_r in the normal skin effect region close to the surface. The Debye surface currents may be written approximately as

$$K_z^D = J_z^D \lambda_D = \sigma_D E_z^D \lambda_D = \sigma\, E_z(r=a,\theta)\, \lambda_D \qquad \text{// dim(K) = amp/m}$$

$$K_\theta^D = J_\theta^D \lambda_D = \sigma_D E_\theta^D \lambda_D = \sigma\, E_\theta(r=a,\theta)\, \lambda_D = 0 \qquad \text{// (D.9.2) and (3.7.0)} \qquad (D.9.5)$$

where we use the Corollary above that $\sigma_D = \sigma$. From (3.7.0) we have $E_\theta = 0$ at the surface so $K_\theta^D = 0$ and we have only the Debye current K_z^D to worry about. Recall that $E_\theta(a,\theta) = 0$ is the second boundary condition (D.2.27) used in Section D.2 to evaluate the a_m and K_m coefficients, and that this condition is itself a topic of interest in Section D.8, and we assume it is valid. We then have,

$$-j\omega\, n(\theta,z) = -J_r(a,\theta,z) + \partial_z K_z^D(\theta,z)$$

$$= -J_r(a,\theta,z) + \partial_z\,[\sigma\, E_z(a,\theta,z)\, \lambda_D] \qquad (D.9.6)$$

Assuming everything has z dependence $e^{j(\omega t - \beta_d z)}$ as in (D.1.4), we replace $\partial_z \rightarrow -j\beta_d$ and then suppress the z arguments to get

$$-j\omega\, n(\theta) = -J_r(a,\theta) - j\beta_d [\sigma E_z(a,\theta) \lambda_D]$$

$$= -\sigma E_r(a,\theta) - j\beta_d [\sigma E_z(a,\theta) \lambda_D]$$

$$= -\sigma E_r(a,\theta)[\, 1 - j\beta_d\lambda_D \frac{E_z(a,\theta)}{E_r(a,\theta)}\,] \quad . \qquad \text{(D.9.7)}$$

If we *assume* that the second term in (D.9.7) can be ignored, we get the charge pumping boundary condition

$$E_r(r=a-\varepsilon,\theta) = (j\omega/\sigma)\, n(\theta) \qquad \textit{(D.2.24)} \qquad \text{(D.9.8)}$$

which in return yields the E fields as stated in (D.2.33) where we see that roughly

$$\frac{E_z(a,\theta)}{E_r(a,\theta)} \sim \left|\frac{\beta}{\beta_d}\right| \quad . \qquad \text{(D.9.9)}$$

Thus, our self-consistent condition for ignoring the second term in (D.9.7) is

$$\beta_d \lambda_D * \frac{E_z(a,\theta)}{E_r(a,\theta)} << 1 \qquad \Leftrightarrow \qquad \beta_d \lambda_D * \left|\frac{\beta}{\beta_d}\right| << 1$$

$$\Leftrightarrow \quad \lambda_D |\beta| << 1 \qquad \Leftrightarrow \qquad \lambda_D \left| e^{j3\pi/4} (\sqrt{2}/\delta)\right| << 1$$

$$\Leftrightarrow \quad (\lambda_D/\delta) << 1 \qquad \Leftrightarrow \qquad (\delta/\lambda_d) >> 1 \qquad \text{// ignore } \sqrt{2}$$

But we know from above that $(\delta/\lambda_d) >> 1$ for any f < 100GHz, so for such f the second term in (D.9.7) can in fact be ignored. We have just proven:

Fact 3: For f < 100 GHz, the Debye surface currents can be ignored in the derivation of the boundary condition $E_r(a-\varepsilon,\theta) = (j\omega/\sigma)\, n(\theta)$. (D.9.10)

(c) Where does surface charge n(θ) come from?

According to our traveling-wave ansatz (D.1.1), all E field related quantities move down a transmission line at v_d as $e^{j(\omega t - \beta_d z)}$. For a low-loss line and a vacuum dielectric, $v_d \approx c$, the speed of light. Therefore,

$$n(\theta,z,t) = n(\theta,0,0)\, e^{j(\omega t - \beta_d z)} \qquad \beta_d = (\omega/v_d) \; . \qquad \text{(D.9.11)}$$

One can ponder and then discard a list of hypotheses concerning where n(θ) "comes from" as it increases and decreases over time at some location z on one of the conductors.

The first hypothesis might be that the individual electrons which make up n(θ) simply travel at v_d in the z direction down the conductor surface, and n(θ) is not fed by any radial currents inside the conductor. In this case one would have $K_z^D(\theta) = v_d\, n(\theta)$. But we know this is not what happens. Apart from the massive energy required to achieve relativistic electron velocities, we know from Appendix N.1 that the

electrons in the Debye layer in fact drift along at something like ~ 1 mm/sec, just as do the regular conduction electrons in the conductor bulk.

The second hypothesis is a variation of the first, where we now allow that the Debye surface current works like any other conduction current, and when one electron moves "to the right" at some point z, a distant electron at z+L moves to the right at nearly the same time, all electrons in a long string moving to the right one position, giving the illusion that a particular electron moved very fast. This does in fact happen, and if it were *all* that happened, again we would have $K_z^D(\theta) = v_d\, n(\theta)$.

Comment : Assume some skin depth $\delta \leq a/10$ so the bulk current is flowing in a sheath of thickness δ just under the conductor surface. The total sheath current is then roughly $2\pi a\delta J_z(a,\theta)$. We can regard this current flow as due to an effective "full surface current" $K_z = J_z\, \delta$. Notice that this "surface current" is different from the "Debye surface current". Based on Fact 2 above, we certainly expect $K_z >> K_z^D$.

A third hypothesis is that somehow $n(\theta,z,t)$ is fed by azimuthal Debye surface currents, or some combination of these along with the z-directed $K_z^D(\theta)$. Our condition (3.7.0) that $E_\theta = 0$ puts a stop to the possibility of feeding by azimuthal Debye surface currents.

What we have learned from Fact 3 is that none of the above hypotheses explains where $n(\theta)$ comes from. The analysis above shows that, although the surface motions of the Debye surface charges do create Debye surface currents, these currents are so small that they play no role in div $J = -\partial_t \rho$ for the Gaussian box shown in red in Fig **. The charge $n(\theta)$ "comes from" inside the wire and is fed by the radial current density J_r just below the surface according to (D.2.24),

$$J_r(r=a-\varepsilon,\theta) = j\omega\, n(\theta) \qquad\qquad n(\theta) = (1/j\omega)\, J_r(a-\varepsilon,\theta) \qquad\qquad (D.9.12)$$

Here is a suggestive picture,

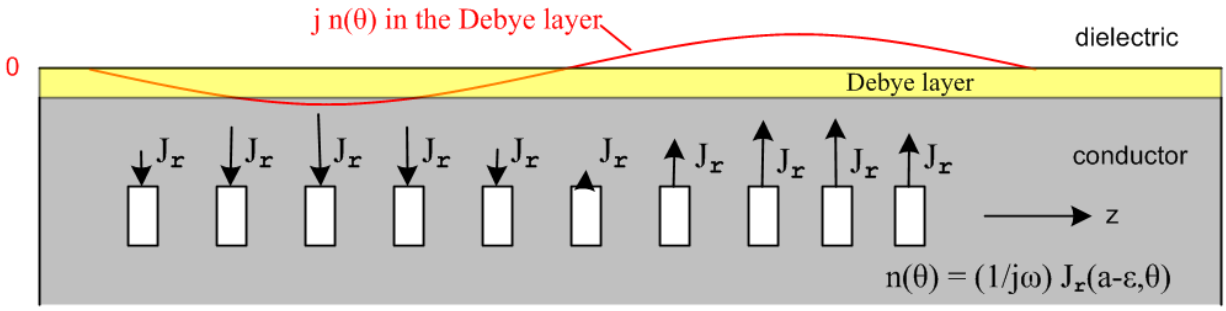

Fig D.7

where the white boxes are little "radial charge pumps" delivering the required J_r needed to feed the changing surface charge $n(\theta)$. Apart from the miniscule K_z^D , charges in $n(\theta)$ don't move in the z direction in this picture, they just appear to be doing that due to the choreographed radial pumping in and out at the wire surface. A wave front of the $n(\theta)$ wave travels at v_d, and this is just a phase velocity. In an analogous situation, in a deep ocean wave the individual particles of water travel in small ellipses and do not travel along with the wave, though there are small scale longitudinal motions due to those ellipses.

Comment: In our transmission line theory, the exterior problem in the dielectric is solved using the capacitor method, from which one learns $n(\theta)$. The boundary condition $J_r(r=a-\varepsilon,\theta) = j\omega\, n(\theta)$ couples this exterior information into the wire interior, allowing one to solve for the fields and currents inside.

In Section 6.5 we show how div **E** = 0 inside the conductor (or div **J** = 0) forces a relationship between J_r and J_z just below the conductor surface. When that relationship (6.5.18) is combined with the charge pumping boundary condition (D.9.8), one finds that

$$E_z(a,\theta) = (-j\omega/\sigma)\,(\beta/\beta_d)\, n(\theta) \qquad \textit{(6.5.19)}$$

or

$$J_z(a,\theta) = (-j\omega)\,(\beta/\beta_d)\, n(\theta)\ . \qquad \text{(D.9.13)}$$

This same result is obtained in a different manner as (6.5.13).

The "full surface current" K_z was defined in a Comment above as $K_z(\theta) = \delta\, J_z(a,\theta)$. Thus,

$$K_z(\theta) = \delta\, J_z(a,\theta) = [\delta\,(-j\omega)\,(\beta/\beta_d)]\, n(\theta) \qquad \text{(D.9.14)}$$

But

$$\frac{\beta}{\beta_d} = \frac{e^{j3\pi/4}\,(\sqrt{2}\,/\delta)}{\omega/v_d}$$

so

$$[\delta\,(-j\omega)\,(\beta/\beta_d)] = \delta\,(-j\omega)\,\frac{e^{j3\pi/4}\,(\sqrt{2}\,/\delta)}{\omega/v_d} = -j\, e^{j3\pi/4}\,\sqrt{2}\; v_d$$

$$= -j\,(j-1)/\sqrt{2}\ *\sqrt{2}\ v_d = (1+j)\, v_d$$

and we end up with

$$K_z(\theta) = (1+j)\, v_d\, n(\theta)$$

$$\mathrm{Re}(K_z(\theta)) = \mathrm{Im}(K_z(\theta)) = v_d\, n(\theta) \qquad \text{(D.9.15)}$$

Once again, this last equation gives the illusion that the surface charge density $n(\theta)$ moves "to the right" at speed v_d to create the real or imaginary part of the full δ-thick surface current K_z. This is the equation that replaces the incorrect equation $K_z^D(\theta) = v_d\, n(\theta)$ which assumes there is no radial charge pumping.

(d) What happens if the dielectric conducts?

Ignoring the Debye surface currents as per section (b) above, if the dielectric has some conductivity σ_d, the charge pumping boundary condition (D.2.23) becomes

$$J_r(a-\alpha,\theta) - J_r(a+\alpha,\theta) = j\omega\, n(\theta)$$

or

$\sigma\, E_r(a-\alpha,\theta) \;-\; \sigma_d E_r(a+\alpha,\theta) \;= j\omega\, n(\theta)$ // this is div J = -jωρ

where $\alpha > 0$ is a tiny distance (ε is already used for dielectric constant). Another boundary condition at the surface is provided by (1.1.47) which says ($\hat{\mathbf{n}}$ points into medium 1 which is the dielectric)

$[\varepsilon_1 E_{n1} - \varepsilon_2 E_{n2}] \;=\; n_{free}$ // this is continuity of D_n at the surface

or

$[\varepsilon_d E_{r1} - \varepsilon_0 E_{r2}] \;=\; n(\theta)$

or

$[\varepsilon_d\, E_r(a+\alpha,\theta) - \varepsilon_0\, E_r(a-\alpha,\theta)] \;=\; n(\theta)$

A seeming third boundary condition is (1.1.48),

$\xi_1 E_{n1} = \xi_2 E_{n2}$

or

$(\varepsilon_d + \sigma_d/j\omega)\, E_{rd} = (\varepsilon_0 + \sigma/j\omega)\, E_r \;\approx (\sigma/j\omega)\, E_r$

or

$(\varepsilon_d + \sigma_d/j\omega)\, E_r(a+\alpha,\theta) \;\approx\; (\sigma/j\omega)\, E_r(a-\alpha,\theta)$.

There seem to be these three boundary conditions at the round wire surface,

$\sigma\, E_r(a-\alpha,\theta) \;-\; \sigma_d E_r(a+\alpha,\theta) \;= j\omega\, n(\theta)$ // modified cpbc from div **J** = -jωρ (D.9.16)

$\varepsilon_d\, E_r(a+\alpha,\theta) \;-\; \varepsilon_0\, E_r(a-\alpha,\theta) \;=\; n(\theta)$ // div **D** = ρ (straddle) (D.9.17)

$(\varepsilon_d + \sigma_d/j\omega)\, E_r(a+\alpha,\theta) \approx\; (\sigma/j\omega)\, E_r(a-\alpha,\theta)$ // $\xi_1 E_{n1} = \xi_2 E_{n2}$ (D.9.18)

but only two of these conditions are independent. For example, multiply (D.9.16) by (-1/jω) to get

$(\sigma_d/j\omega) E_r(a+\alpha,\theta) \;-\; (\sigma/j\omega)\, E_r(a-\alpha,\theta) \;=\; -\, n(\theta)$.

Adding this to (D.9.17) then gives

$(\varepsilon_d + \sigma_d/j\omega)\, E_r(a+\alpha,\theta) \;-\; (\sigma/j\omega)\, E_r(a-\alpha,\theta) \;=\; 0$

which is in fact the same as (D.9.18).

When we solve the "capacitor problem" as in Section 6.5 a) to obtain n(θ) on the conductor surface, we are using (D.9.17) with the assumption that $E_r(a+\alpha,\theta) >> E_r(a-\alpha,\theta)$. Typically one just says that in a good conductor $E_r(a-\alpha,\theta) = 0$ and then $n(\theta) = \varepsilon_d\, E_r(a+\alpha,\theta)$. That is fine, but it is not clear what happens to (D.9.16) above. The first term is the product of a large quantity σ times a small quantity $E_r(a-\alpha,\theta)$ so can be the same size as the other terms in the equation.

The resolution is provided by the discussion in Section 1.5 (c) where we encountered the equation (1.5.17)

$$n_c(\mathbf{x},\omega) = (\xi_1/\varepsilon_1)\, n_s(\mathbf{x},\omega) \,. \qquad (1.5.17)$$

which in our current context (1 = dielectric) becomes

$$n_c(\theta) = (\xi_d/\varepsilon_d)\, n(\theta) \,. \qquad (D.9.19)$$

In that discussion it is noted that $n(\theta)$ is the actual free surface charge density, whereas $n_c(\theta)$ is a related "transport charge density" having the same dimensions as $n(\theta)$. If we multiply (D.9.16) and (D.9.17) by (ξ_d/ε_d), our (redundant) triplet of boundary conditions becomes,

$$\begin{aligned}
&1 \quad \sigma\,(\xi_d/\varepsilon_d)\, E_r(a\text{-}\alpha,\theta) \;-\; \sigma_d\,(\xi_d/\varepsilon_d)\, E_r(a\text{+}\alpha,\theta) \;=\; j\omega\, n_c(\theta)\\
&2 \quad \xi_d\; E_r(a\text{+}\alpha,\theta) \;-\; (\xi_d/\varepsilon_d)\,\varepsilon_0\, E_r(a\text{-}\alpha,\theta) \;=\; n_c(\theta)\\
&3 \quad \xi_d E_r(a\text{+}\alpha,\theta) \approx \xi\, E_r(a\text{-}\alpha,\theta) \,. \qquad (D.9.20)
\end{aligned}$$

We now use last of these three equations to eliminate $E_r(a+\alpha,\theta)$ in the first, which then becomes

$$\begin{aligned}
&\sigma\,(\xi_d/\varepsilon_d)\, E_r(a\text{-}\alpha,\theta) \;-\; \sigma_d\;(\xi/\varepsilon_d)\, E_r(a\text{-}\alpha,\theta) \;=\; j\omega\, n_c(\theta)\\
&\text{or}\\
&[\,\sigma\,\xi_d - \sigma_d\,\xi\,]/\varepsilon_d\, E_r(a\text{-}\alpha,\theta) \;=\; j\omega\, n_c(\theta)\\
&\text{or}\\
&[\,\sigma\,(\varepsilon_d + \sigma_d/j\omega) - \sigma_d\,(\varepsilon_0 + \sigma/j\omega)\;]/\varepsilon_d\, E_r(a\text{-}\alpha,\theta) \;=\; j\omega\, n_c(\theta)\\
&\text{or}\\
&[\,(\sigma\,\varepsilon_d - \sigma_d\varepsilon_0)]/\varepsilon_d\, E_r(a\text{-}\alpha,\theta) \;=\; j\omega\, n_c(\theta) \qquad \text{// two large terms cancelled}\\
&\text{or}\\
&[\,(\sigma - \sigma_d(\varepsilon_0/\varepsilon_d)\,]\, E_r(a\text{-}\alpha,\theta) \;=\; j\omega\, n_c(\theta)
\end{aligned}$$

We now assume that ε_0 (conductor) and ε_d (dielectric) are the same order of magnitude, and we assume that, even though the dielectric conducts, we still have $\sigma >> \sigma_d$. The last equation then reads

$$E_r(a\text{-}\alpha,\theta) \;=\; (j\omega/\sigma)\, n_c(\theta) \;=\; (j\omega/\sigma)\,(\xi_d/\varepsilon_d)\, n(\theta) \qquad (D.9.21)$$

$$E_r(a\text{-}\alpha,m) = (j\omega/\sigma)\,(\xi_d/\varepsilon_d)\, N_m \,. \qquad \text{// partial waves} \qquad (D.9.22)$$

The above are the **"modified" charge pumping boundary conditions** which replaces (D.2.24) and (D.2.25) for a mildly conducting dielectric

$$E_r(r\text{=}a,\theta) = (j\omega/\sigma)\, n(\theta) \,. \qquad \textit{(D.2.24)}$$

$$E_r(r\text{=}a,m) = (j\omega/\sigma)\, N_m \,. \qquad \textit{(D.2.25)}$$

How then does $\sigma_d \neq 0$ alter the E field results summarized in box (D.2.33)? If we indicate $\sigma_d = 0$ quantities with no prime and $\sigma_d > 0$ quantities with a prime, then tracing through the development we find that

$$n(\theta) \rightarrow (\xi_d/\varepsilon_d)\, n(\theta) \equiv n'(\theta)$$

$$N_m \rightarrow (\xi_d/\varepsilon_d)\, N_m \equiv N'_m \qquad \text{but } \eta_m \equiv N_m/N_0 \text{ remains the same}$$

$$K_m \rightarrow (\xi_d/\varepsilon_d)\, K_m \equiv K'_m$$

$$a_m \rightarrow (\xi_d/\varepsilon_d)\, a_m \equiv a'_m$$

$$\beta_d = \omega \sqrt{\mu_d}\sqrt{\varepsilon_d} \rightarrow \omega \sqrt{\mu_d}\sqrt{\xi_d} \equiv \beta_d' \qquad \text{so} \qquad \beta_d' = \sqrt{\xi_d/\varepsilon_d}\, \beta_d \tag{D.9.23}$$

The normalization condition (D.2.31)

$$I = 2\pi\omega\, (a/\beta_d)\, N_0 \tag{D.2.31}$$

becomes (where I' is the total current in the wire with $\sigma_d > 0$)

$$I' = 2\pi\omega\, (a/\beta'_d)\, N_0' = 2\pi\omega\, (a/\beta_d) \sqrt{\varepsilon_d/\xi_d}\, (\xi_d/\varepsilon_d) N_0 = 2\pi\omega\, (a/\beta_d) \sqrt{\xi_d/\varepsilon_d}\, N_0 = \sqrt{\xi_d/\varepsilon_d}\; I\,. \tag{D.9.24}$$

Then for example the E_z field becomes (see below (D.2.32)),

$$E'_z(r,m) = -j(\beta'/\beta'_d)\, \frac{K'_m}{2}\, J_m(x) = -j(\beta'/\beta'_d)\, (j\omega/2\sigma)\, N'_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] J_m(x)$$

$$= -j(\beta'/\beta'_d)\, (j\omega/2\sigma)\, \eta_m\, N'_0 \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] J_m(x)$$

$$= -j(\beta'/\beta'_d)\, [(j/4)\, (a\beta'_d)\, \eta_m\, I'\, R_{dc}] \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] J_m(x)$$

$$= (1/4)\, \eta_m\, I'\, R_{dc}\, (a\beta') \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right]$$

and for the E_r field,

$$E'_r(r,m) = a'_m\, x^{-1}\, J_m(x) + \frac{K'_m}{2}\, J_{m+1}(x)$$

$$= [(j\omega/2\sigma)\, N'_m] \left\{ 2m\, \frac{1}{J_{m-1}(x_a)}\, x^{-1}\, J_m(x) + \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] J_{m+1}(x) \right\}$$

$$= (j/4)\,(a\beta'_d)\,\eta_m\, I'\, R_{dc} \quad \{ \frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m+1}(x)}{J_{m-1}(x_a)} \}$$

$$= (j/4)\,(a\beta'_d)\,\eta_m\, I'\, R_{dc} \quad \{ \frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)} \}.$$

The results (D.2.33) then become:

Second summary of the E field solutions : $\quad R_{dc} = \frac{1}{\sigma\pi a^2} \quad \beta'^2 = \beta^2 - \beta'^2_d \quad$ (D.9.25)

$\beta'_d = \sqrt{\xi_d/\varepsilon_d}\ \beta_d$ where $\beta_d = \omega\sqrt{\mu_d\varepsilon_d}$ $\qquad I' = \sqrt{\xi_d/\varepsilon_d}\ I$

$E'_z(r,m) = (1/4)\,\eta_m\, I'\, R_{dc}\ (a\beta')\, f_m \qquad f_m = [\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}] \qquad x = \beta' r$

$E'_r(r,m) = (j/4)\,\eta_m\, I'\, R_{dc}\ (a\beta'_d)\, g_m \qquad g_m = [\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}] \qquad x_a = \beta' a$

$E'_\theta(r,m) = (1/4)\,\eta_m\, I'\, R_{dc}\ (a\beta'_d)\ h_m \qquad h_m = [\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}]$

Conclusion: Turning on σ_d causes the total current to "increase" from I to $I' = \sqrt{\xi_d/\varepsilon_d}\ I$, which seems intuitive since the total current now has to feed the conductance between the conductors. However, the E field solutions have exactly the same form they had when $\sigma_d = 0$ when expressed in terms of the new larger current I' and the new β'_d. This β'_d differs from the β_d for $\sigma_d = 0$, and this then gets mapped into a slight difference in β' since $\beta'^2 = \beta^2 - \beta'^2_d$.