Appendix D and Smythian Forms? PhL 10.13.13

Question: Could the method of Smythian Forms have been used in Appendix D? That would have required the atomic forms for the vector Helmholtz equation in cylindrical coordinates. Most of my work involved atomic forms for the Laplace equation, but I know I did some Helmholtz.

Yes, Stak 7.12 is all about this. The atomic forms are e^{ikz} in the z direction, $e^{im\varphi}$ in the φ direction, and J_ν in the r direction. See for example Stak p 282 for some Smythian forms like 7.208 and 7.209. You see there the atomic forms just mentioned. Stak's λ is my β^2 . However, Stak always seems to deal with a *scalar* Helm equation, whereas I am dealing with the *vector* one. Perhaps this is where those "vector harmonics" come into play (Carleton DeTar stuff). I just found my doc on this subject in he physics/E&M area. Search term is "vector spherical harmonics" and it is a truly big mess.

Meanwhile, M&S also talk about the vector Helmholtz equation a lot, all of their Chapter 5.

Digression: On page 171 they talk about ″the Bessel Wave Equation" which is more general that what I got in Appendix D. This is shown in M&S top of p 171 where they have a term $-p^2/z^2$ which I do not have. They don't say where this equation is coming from? If I look at M&S page 15 where they talk separation of Helm in cylindricals, their radial equation is like mine and does not have the extra term. Sphericals also does not have this term. But I see this extra term appearing in **Parabolic Coordinates** on page 36 of M&S where μ is the variable instead of r. So parabolic is the only one! I confirm this in a Google Book of Voon.

So the reason I don't have these Bessel wave functions is that I am not using parabolic coordinates. These Bessel wave functions have NOTHING to do with what I am doing here.

But hold the phone! M&S page 139 talk about the "vector Helm equation" in cylindricals on page 139. There κ^2 is my β^2, the H parameter. But OK, same thing as in my App D, no Bessel wave stuff, all OK.

So yes I guess I could have used a Smythian form method, something like this

$$E_r = \Sigma_m \left(Ae^{ikz} + Be^{-ikz} \right) \quad ???$$

Scalar Helmholtz:

I am confused, let's back up and look at **spherical** coordinates and the **Laplace** equation. M&S shows this on page 26 where they use (r,θ,ψ). They show the separated equations and the ψ equation has harmonics that are obviously $e^{im\psi}$. The radial gives powers like r^ℓ and the θ equation gives associated Legendres, and that is where the atomic forms come from that I often use. These are oscillatory in θ and ψ and expo in r.

Next, consider **spherical** and **Helmholtz** page 27. Only the radial equation has changed and instead of powers of r we get some Bessel functions $J_\nu(kr)$. We are doing scalar functions only here.

Next, consider **cylindrical** and **Laplace**. The ψ and z equations are trivial givint $e^{im\psi}$ and e^{ikz} and then the action is in the radial which is regular Bessel $J_\nu(kr)$again.

Next, consider **cylindrical** and **Helmholtz** p 15. Now we get $e^{im\psi}$ still. The Z equation is also trivial but has a shifted k^2 value which is my shifted β thing, fine. So still $e^{ik'z}$ atom. And then $J_m(kr)$ I think.

Now how does this change when we go to

Vector Helm:

M&S don't have a section on Vector Laplace, only Vector Helm in Section V, so to get Laplace you have to set their $\kappa^2 = 0$ to get these results

Consider **spherical** and vector **Helmholtz.** M&S first write out the three equations on page 141, but you don't get "separated equations" in the general case. The three fields are cross coupled. They do discuss then some special cases. MY special case is

$$\mathbf{E}(r,\varphi,z,t) = e^{j(\omega t-\beta_d z)}\ \mathbf{E}(r,\varphi)$$

$$\mathbf{E}(r,\varphi,z,\omega) = \delta(\omega-\omega_1)\ e^{-j\beta_d z}\ \mathbf{E}(r,\varphi)$$

so I guess the special case is

$$\mathbf{E} = e^{-j\beta_d z}\ \mathbf{E}(r,\varphi)$$

where I have all three directions active. M&S don't treat this special case, only I do! First let's write out the general M&S equations with full $E_i(r,\varphi,z)$ arguments:

$$[\partial_r^2 + (1/r)\partial_r + (1/r^2)\partial_\varphi^2 + \partial_z^2]\ E_r - (2/r^2)\partial_\varphi E_\varphi + (\kappa^2 - 1/r^2)\ E_r = 0$$

$$[\partial_r^2 + (1/r)\partial_r + (1/r^2)\partial_\varphi^2 + \partial_z^2]\ E_\varphi + (2/r^2)\partial_\varphi E_r + (\kappa^2 - 1/r^2)\ \}\ E_\varphi = 0$$

$$[\partial_r^2 + (1/r)\partial_r + (1/r^2)\partial_\varphi^2 + \partial_z^2]\ E_z + \kappa^2 E_z = 0$$

All three variables appear in each equation. We don't get "separated equations" at all, so we don't have "atomic forms" in the usual sense!!! This is true even if $\kappa = 0$ and we are talking vector Laplace.

So one conclusion is this: You cannot solve the vector Helmholtz equation using the Smythian form method with "atomic forms" which are geared to cylindrical coordinates. There are no atomic forms because there is no separation.

However, for the cylindrical geometry I know that $e^{im\varphi}$ form a complete set of functions with integer m, since $f(\varphi+2\pi) = f(\varphi)$ for any field we might encounter in this geometry. These are NOT atoms for our situation, but they are viable functions to expand ON. When we go ahead and do that, we end up with what I did in Appendix D.

Fact 1: The scalar Helmholtz equation IS separable in cylindrical coordinates and you CAN write atomic forms of the form $e^{im\psi}$, $e^{ik'z}$ and $J_\nu(kr)$.

Fact 2: The vector Helmholtz equation is NOT separable (not even simply- = R-separable) in cylindrical coordinates, so you CANNOT write down atomic forms.

Fact 3: You can always expand a function of azimuthal φ on the $e^{im\varphi}$ as I have done in Appendix D. These don't have to be part of the atoms of the problem.

Fact 4: The vector spherical harmonics of the vector Laplace equation exist and are useful in certain problems as discussed in green Jackson Chapter 16, but I don't think they have any application in my current discussion. They are defined by

$$\mathbf{X}_{\ell m} = \mathbf{r} \times \nabla\, Y_{\ell m} \qquad \mathbf{Y}_{\ell m} = \hat{\mathbf{r}}\, Y_{\ell m} \qquad \mathbf{Z}_{\ell m} = r\nabla\, Y_{\ell m}$$

or alternately,

$$\mathbf{X}^1_{\ell m} = k_\ell\, \mathbf{X}_{\ell m} = k_\ell\, \mathbf{r} \times \nabla\, Y_{\ell m} = i\, k_\ell\, \mathbf{L}\, Y_{\ell m}$$
$$\mathbf{X}^2_{\ell m} = \mathbf{Y}_{\ell m} = \hat{\mathbf{r}}\, Y_{\ell m}$$
$$\mathbf{X}^3_{\ell m} = k_\ell\, \mathbf{Z}_{\ell m} = k_\ell\, r\, \nabla Y_{\ell m}$$

$$\int d\Omega\ \mathbf{X}^{n'}_{\ell' m'}(\theta,\phi)^* \bullet \mathbf{X}^n_{\ell m}(\theta,\phi) = \delta_{\ell',\ell}\, \delta_{m',m}\, \delta_{n',n} \qquad \text{orthogonality}$$

$$\Sigma_{\ell mn}\ [\mathbf{X}^n_{\ell m}{}^*(\theta,\phi)]_i\ [\ \mathbf{X}^n_{\ell m}(\theta',\phi')\ ]_{i'} = \delta_{i,i'}\ \langle\theta\phi|\theta'\phi'\rangle = \delta_{i,i'}\ \delta(\Omega - \Omega') \quad \text{completeness}$$

$$\mathbf{A}(\theta,\phi) = \Sigma_{\ell mn}\ A_{\ell mn}\, \mathbf{X}^n_{\ell m}(\theta,\phi) \qquad (1)$$

I don't think these vector harmonics satisfy the vector Laplace equation. My paper shows:

$$\nabla^2\, \mathbf{X}^1_{\ell m} = -\,\ell(\ell+1)/r^2\, \mathbf{X}^1_{\ell m} \qquad \Rightarrow \qquad [\nabla^2 + \ell(\ell+1)/r^2]\, \mathbf{X}^1_{\ell m} = 0$$

$$\nabla^2\, \mathbf{X}^2_{\ell m} = (2/\, k_\ell\, r^2)\, \mathbf{X}^3_{\ell m} - [\ell(\ell+1) + 2]\, /r^2\, \mathbf{X}^2_{\ell m}$$

$$\nabla^2\, \mathbf{X}^3_{\ell m} = 2\, k_\ell\, \ell(\ell+1)/r^2\, \mathbf{X}^2_{\ell m} - \ell(\ell+1)/\, r^2\, \mathbf{X}^3_{\ell m}$$

None of these is a vector Laplace or vector Helmholtz equation. These are vector functions onto which you can expand **E** and **B** in complete generality and then you can talk about a particular multipole level with some particular $\boldsymbol{\ell}$ value.

Fact 5: I just added a Comment at the end of App D.1 (b) which states Facts 1,2,3 above. I think this is a good addition to the paper.

So I think I have done due diligence on this side show topic.