Old Appendix D 10.13.13

I had my new Appendix D in a separate file for a long time, but today I decided to install it into the main doc since I guess it is reasonably stable. Here I store the App D that got replaced.

Appendix D: The General Electric Field in a Round Wire

In this Appendix we present a somewhat lengthy calculation of the electric field (and therefore the current) in a round wire *without* assuming that such fields are symmetrical about the axis.

The conclusions one can draw from the analytic solution given below are supportive of the general discussion elsewhere in this monograph:

1. The E_r and E_φ field components are *very small*. In fact, they are smaller than E_z by the factor (a/λ), where a = wire radius, and λ = wavelength of the wave on a transmission line containing the wire. This fraction is always assumed small in any analysis of a transmission line.

2. It is possible to maintain the condition $E_\varphi = 0$ on the wire surface. This condition is consistent with the idea that the wire surface is an electrical equipotential at any constant z.

3. We get an explicit formula for the *surface impedance*. We find that it is non-uniform around the boundary of the wire cross section, and that J_z is non-uniform inside the wire. This result is true even in the DC limit of a transmission line.

4. We suggest a method for computing the E fields in the low frequency limit of a transmission line consisting of round wires. First, solve the "electrostatic" problem illustrated in Chapter 6. Knowing the potential φ, compute the radial electric field at the surface of the conductors. From this compute the surface charge density n(φ) as a function of azimuth around the wire. Then, as described below, compute the moments N_m (or η_m) of this charge distribution, and use the formulas below to find the electric field.

This is a highly technical appendix. The reader is invited to inspect the boxed results and comments at the end.

D.1 The General Method and Solution for E_z

The starting point for the calculation is the Helmholtz equation (1.6.2) for the E field inside the wire,

$$(\nabla^2 + \beta^2) \mathbf{E}(r,\varphi, z, t) = 0 . \qquad (D.1.1)$$

Here we use cylindrical coordinates (r,φ,z), as appropriate for a straight round wire. The first step is to expose the assumed t and z dependence, and in doing so, define the field $\mathbf{E}(r,\varphi)$:

$$\mathbf{E}(r,\varphi,z,t) = e^{j(\omega t - \beta_d z)}\, \mathbf{E}(r,\varphi) . \qquad (D.1.2)$$

Here, symbol β_d stands for the value that parameter $\beta = \omega\sqrt{\mu\xi}$ takes in the dielectric. We reserve the symbol β with no subscript to mean β inside the metal of our wire. The form shown in (D.1.2) is a simple wave traveling down a transmission line in the +z direction. We assume that one of the conductors of this transmission line is our round wire, while the other conductor is (other conductors are) unspecified.

One can regard (D.1.2) as an assumed variable-separated form for a solution, an "ansatz". Our second ansatz is that $E_\varphi(r=a,\varphi) = 0$ which means at any z = constant slice, the circular wire perimeter is an equipotential. If a consistent solution to the field equations can be found with these assumptions, they are justified *de facto*.

(a) Partial Wave Expansions

The next step is to do a "partial wave expansion" (that is, a complex Fourier series expansion) of $\mathbf{E}(r,\varphi)$ in terms of "azimuthal harmonics" $e^{jm\varphi}$, so that the variable φ is replaced with the partial wave index m:

$$\mathbf{E}(r,\varphi) = \sum_{m=-\infty}^{\infty} \mathbf{E}(r,m)\, e^{jm\varphi} \qquad \Leftrightarrow \qquad \mathbf{E}(r,m) = (1/2\pi) \int_{-\pi}^{\pi} d\varphi\, \mathbf{E}(r,\varphi)\, e^{-jm\varphi} \tag{D.1.3}$$

Since the usual complex part has been extracted in (D.1.2), we assume that $\mathbf{E}(r,\varphi)$ is real so the right euqation above says that $\mathbf{E}(r,-m) = \mathbf{E}(r,m)^*$. We can then re-express the above as,

$$\mathbf{E}(r,\varphi) = \mathbf{E}(r,0) + \sum_{m=1}^{\infty} [\, \mathbf{E}(r,m)e^{jm\varphi} + \mathbf{E}(r,-m)e^{-jm\varphi}] = \mathbf{E}(r,0) + \sum_{m=1}^{\infty} [\, \mathbf{E}(r,m)e^{jm\varphi} + [\mathbf{E}(r,m)e^{jm\varphi}]^*]$$

$$= \mathbf{E}(r,0) + 2 \sum_{m=1}^{\infty} \mathrm{Re}\{\, \mathbf{E}(r,m)e^{jm\varphi} \}$$

$$= \mathbf{E}(r,0) + 2 \sum_{m=1}^{\infty} [\, \mathrm{Re}(\mathbf{E}) + j\, \mathrm{Im}(\mathbf{E})]\, [\, \cos(m\varphi) + j \sin(m\varphi)]$$

$$= \mathbf{E}(r,0) + 2 \sum_{m=1}^{\infty} [\mathrm{Re}(\mathbf{E})\cos(m\varphi) - \mathrm{Im}(\mathbf{E}) \sin(m\varphi)] + 2j \sum_{m=1}^{\infty} [\, \mathrm{Im}(\mathbf{E})\cos(m\varphi) + \mathrm{Re}(\mathbf{E}) \sin(m\varphi)]$$

Since $\mathbf{E}(r,\varphi)$ is assumed real, the second sum must vanish for all r and φ, giving this final result,

$$\mathbf{E}(r,\varphi) = \mathbf{E}(r,0) + 2 \sum_{m=1}^{\infty} [\mathrm{Re}\{\mathbf{E}(r,m)\}\cos(m\varphi) - \mathrm{Im}\{\mathbf{E}(r,m)\} \sin(m\varphi)] \,. \tag{D.1.4}$$

In analogy with (D.1.2) and the discussion of Section 1.7 regarding complex functions, we define a complex surface (r=a) charge density $n(\varphi,z,t)$ which has the following ansatz variable-separated form,

$$n(\varphi,z,t) = e^{j(\omega t - \beta_d z)}\, n(\varphi) \tag{D.1.5}$$

where n(φ) is real. We then expand n(φ) as in (D.1.3) and (D.1.4),

$$n(\varphi) = \sum_{m=-\infty}^{\infty} N_m e^{jm\varphi} \qquad \Leftrightarrow \qquad N_m = (1/2\pi) \int_{-\pi}^{\pi} d\varphi\, n(\varphi)\, e^{-jm\varphi} \ . \tag{D.1.6}$$

$$n(\varphi) = \mathbf{N_0} + 2 \sum_{m=1}^{\infty} [\mathrm{Re}\{N_m\}\cos(m\varphi) - \mathrm{Im}\{N_m\}\sin(m\varphi)] \tag{D.1.7}$$

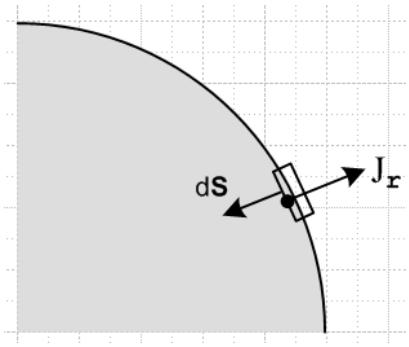
Since n(φ) is real, $N_m = N_{-m}^*$. These N_m are the "moments" of the surface charge distribution, and their complex nature keeps track of cos(mφ) versus sin(mφ) components.

(b) Charge Pumping Boundary Condition

The reason we are so interested in n(φ) is that it acts as a driving source of the radial electric field in the wire. Recall the equation of continuity (1.1.8),

$$\mathrm{div}J = -j\omega\rho \qquad \Leftrightarrow \qquad \int_S dS \bullet J = -j\omega \int_V dV\, \rho \ . \qquad \text{// divergence theorem} \tag{D.1.8}$$

When applied to a thin box of radial area dS straddling the wire surface,



one finds that $\int_S d\mathbf{S}\bullet\mathbf{J} = -J_r(r=a,\varphi)dS$ and $\int_V dV\, \rho = n(\varphi)\, dS$ so that

$$J_r(r=a,\varphi) = j\omega\, n(\varphi) \ . \tag{D.1.9}$$

We assume that there is no current outside the wire to get this result. Since **J** = σ**E**, this is really a boundary condition on the radial electric field,

$$E_r(r=a,\varphi) = (j\omega/\sigma)\, n(\varphi) \ . \tag{D.1.10}$$

Since we have parallel partial wave expansions for both sides, we can rewrite (D.1.10) in m space as

$$E_r(r=a,m) = (j\omega/\sigma)\, N_m \ . \tag{D.1.11}$$

Thus, the radial electric field must have a certain value at the r=a boundary in each partial wave. And the value it must have is determined by the moment of the charge distribution.

By way of interpretation, the surface charge of a transmission line is "pumped" by the radial current in the wire. Since div**J** = 0 inside the wire, this radial current is accompanied by the usual longitudinal current one expects to find inside the conductors of a transmission line.

(c) The E_z Solution

Here is the Helmholtz equation (D.1.1) for E_z as in (D.1.2) (cylindrical coordinates r,φ,z)

$$[\partial_r^2 + (1/r)\,\partial_r + (1/r^2)\,\partial_\varphi^2 + \partial_z^2 + \beta^2\ ]\ e^{j(\omega t-\beta_d z)} E_z(r,\varphi) = 0\ . \qquad \text{(D.1.12)}$$

Inserting the expansion (D.1.3) and moving the m sum to the left gives

$$\sum_{m=-\infty}^{\infty} [\partial_r^2 + (1/r)\,\partial_r + (1/r^2)\,\partial_\varphi^2 + \partial_z^2 + \beta^2\ ]\ e^{j(\omega t-\beta_d z)} E_z(r,m)\ e^{jm\varphi} = 0\ . \qquad \text{(D.1.13)}$$

We can then make the obvious replacements $\partial_z = -j\beta_d$ and $\partial_\varphi = +jm$ to get,

$$\sum_{m=-\infty}^{\infty} \{\ [\partial_r^2 + (1/r)\,\partial_r - m^2\,(1/r^2) - \beta_d^2 + \beta^2\ ]\ e^{j(\omega t-\beta_d z)} E_z(r,m)\ \}\ e^{jm\varphi} = 0\ . \qquad \text{(D.1.14)}$$

Due to the completeness of functions $e^{jm\varphi}$ on the interval $(-\pi.\pi)$, we conclude that $\{\ \} = 0$, or

$$[\partial_r^2 + (1/r)\,\partial_r - m^2\,(1/r^2) - \beta_d^2 + \beta^2\ ]\ e^{j(\omega t-\beta_d z)} E_z(r,m)\ \ =\ 0\ . \qquad \text{(D.1.15)}$$

[*I.e.*, apply $\int_{-\infty}^{\infty} d\varphi\ e^{-jm'\varphi}$ to both sides of (D.1.14) and use $\int_{-\pi}^{\pi} d\varphi\ e^{j(m-m')\varphi} = 2\pi\,\delta_{m,m'}$.]

Next, multiply both sides of (D.1.15) by $r^2\ e^{-j(\omega t-\beta_d z)}$ to get,

$$[r^2\partial_r^2 + r\,\partial_r - m^2 + r^2\,(\beta^2 - \beta_d^2)]\ E_z(r,m)\ \ =\ 0$$

or

$$[r^2\partial_r^2 + r\,\partial_r + (r^2\,\beta'^2\ - m^2)]\ E_z(r,m)\ \ =\ 0 \qquad \text{(D.1.16)}$$

where

$$\beta'^2 = \beta^2 - \beta_d^2\ . \qquad \text{(D.1.17)}$$

In a conductor, β is huge compared to the dielectric β_d, so we could ignore the distinction between β and β'. Setting $x = \beta' r$ we find $\partial_r = \beta'\partial_x$ and then $r\partial_r = x\partial_x$ and so on so that (D.1.16) reads

$$[x^2\partial_x^2 + x\,\partial_x + (x^2 - m^2)]\,E_z(x/\beta',m) = 0 . \qquad x = \beta' r \qquad \text{(D.1.18)}$$

This is Bessel's equation [Spiegel 24.1] and the solution subject to the condition that E_z be finite at r = 0 is $E_z(x/\beta',m) = C_{zm}\,J_m(x)$ or

$$E_z(r,m) = C_{zm}\,J_m(\beta' r) \qquad \text{(D.1.19)}$$

where C_{zm} is an arbitrary constant for each partial wave m.

Equation (D.1.19) is consistent with (2.1.22). In Section 2.1 we dealt only with the m=0 partial wave, which embodies the symmetrical part of the problem. Also, we assumed constant z behavior so $\beta_d = 0$ and the question of β versus β' never arose.

D.2 The Solutions for E_r and E_φ

In Cartesian coordinates, the Helmholtz equation (D.1.1) can be written $(\nabla^2 + \beta^2)E_i = 0$ for i = 1,2,3 meaning x,y,z. The point is that $(\nabla^2\mathbf{E})_i = \nabla^2(E_i)$. We quietly used this fact for E_z in the previous section. For non-Cartesian coordinates like r and φ things are more complicated (see for example Morse and Feshbach Vol I p 116, Moon and Spencer p 139, or do a web search on "vector Laplacian"; the author's Tensor Analysis document, Sections 13, 14 and 15, derives these results for arbitrary coordinate systems):

$$\mathbf{E} = E_r\hat{\mathbf{r}} + E_\varphi\hat{\boldsymbol{\varphi}} + E_z\hat{\mathbf{z}} \qquad \text{(D.2.1)}$$

$$(\nabla^2\mathbf{E})_r = \nabla^2 E_r - (2/r^2)\,\partial_\varphi E_\varphi - (1/r^2)\,E_r \qquad \text{(D.2.2a)}$$
$$(\nabla^2\mathbf{E})_\varphi = \nabla^2 E_\varphi + (2/r^2)\,\partial_\varphi E_r - (1/r^2)\,E_\varphi \qquad \text{(D.2.2b)}$$
$$(\nabla^2\mathbf{E})_z = \nabla^2 E_z \qquad \text{(D.2.2c)}$$

where

$$\nabla^2 = (1/r)\partial_r(r\partial_r) + (1/r^2)\partial_\varphi^2 + \partial_z^2 = \partial_r^2 + (1/r)\partial_r + (1/r^2)\partial_\varphi^2 + \partial_z^2 . \qquad \text{(D.2.3)}$$

Here is a summary of vector differential operators in cylindrical coordinates taken from Morse and Feshbach above, where the last line corresponds to the above discussion:

$$\begin{aligned}
&\text{grad } \psi = \mathbf{a}_r\frac{\partial\psi}{\partial r} + \mathbf{a}_\phi\frac{1}{r}\frac{\partial\psi}{\partial\phi} + \mathbf{a}_z\frac{\partial\psi}{\partial z}\\
&\text{div } \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}\\
&\text{curl } \mathbf{A} = \mathbf{a}_r\left(\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \mathbf{a}_\phi\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \mathbf{a}_z\left(\frac{1}{r}\frac{\partial}{\partial r}rA_\phi - \frac{1}{r}\frac{\partial A_r}{\partial\phi}\right)\\
&\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}\\
&\nabla^2\mathbf{A} = \mathbf{a}_r\left[\nabla^2 A_r - \frac{A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\phi}{\partial\phi}\right] + \mathbf{a}_\phi\left[\nabla^2 A_\phi - \frac{A_\phi}{r^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial\phi}\right] + \mathbf{a}_z\nabla^2 A_z
\end{aligned} \qquad \text{(D.2.4)}$$

(a) The E_r Solution

As noted above, it is a characteristic of curvilinear coordinate systems that the Laplacian operator applied to a vector quantity involves a cross coupling between field components which does not occur in Cartesian coordinates. In our case, this affects the E_r and E_φ field components, but not E_z. Here is the Helmholtz equation (D.1.1) for $E_r(r,\varphi,z)$ using (D.2.2a),

$$\nabla^2 E_r - (2/r^2)\,\partial_\varphi E_\varphi - (1/r^2)\,E_r + \beta^2 E_r = 0 \quad . \tag{D.2.5}$$

The fact that E_φ is cross-coupled with E_r is a minor inconvenience. We eliminate it by using the condition that div**E** = 0, which in cylindrical coordinates appears as in (D.2.4) times r,

$$\partial_r (r\,E_r) + \partial_\varphi E_\varphi + r\,\partial_z E_z = 0 \quad . \tag{D.2.6}$$

Expanding the ∇^2 operator as in (D.2.3), and eliminating $\partial_\varphi E_\varphi$ from (D.2.6) , we get:

$$[\partial_r^2 + (1/r)\partial_r + (1/r^2)\partial_\varphi^2 + \partial_z^2]E_r + (2/r^2)\,[\partial_r (r\,E_r) + r\,\partial_z E_z] - (1/r^2)\,E_r + \beta^2 E_r = 0$$

or

$$\partial_r^2 E_r + (3/r)\,\partial_r E_r + (1/r^2)\,E_r + (1/r^2)\,\partial_\varphi^2 E_r + \partial_z^2 E_r + \beta^2 E_r = (-2/r)\,\partial_z E_z \,. \tag{D.2.7}$$

We got rid of E_φ, but now we are stuck with E_z on the right. Actually, we are happy to have this factor on the right, because this is how the longitudinal current in the wire gets coupled into the radial current which feeds the surface charge. Recall that

$$\mathbf{E}(r,\varphi,z,t) = e^{j(\omega t - \beta_d z)} \sum_{m=-\infty}^{\infty} \mathbf{E}(r,m)\, e^{jm\varphi} \quad . \tag{D.1.2), (D.1.3}$$

As in the previous section, we insert into (D.2.7) the above expansion for the various E components, replace $\partial_\varphi^2 = -m^2$ and $\partial_z = -j\beta_d$, use completeness of the $e^{jm\varphi}$, install (D.1.19) for $E_z(r,m)$, and multiply by r^2 -- with β'^2 as in (D.1.17) -- to get,

$$[r^2\partial_r^2 + 3r\partial_r + (1-m^2) + r^2\,\beta'^2]\,E_r(r,m) = 2j\,\beta_d\,r\,C_{zm}\,J_m(\beta' r) \,. \tag{D.2.8}$$

In order to get the left side into something recognizable, we define

$$E_r(r,m) = x^{-1}\,f_m(x) \tag{D.2.9}$$

where x is a dimensionless radial variable which will play a major role in the following,

$$x \equiv \beta' r \qquad \text{and} \qquad x_a \equiv \beta' a \quad . \tag{D.2.10}$$

Then (D.2.8) becomes

$$[x^2\partial_x^2 + 3x\partial_x + (1-m^2) + x^2]\ \{ x^{-1} f_m(x)\} = 2j\ (\beta_d/\beta')\ xC_{zm}\ J_m(x)$$

or

$$x\ [x^2\partial_x^2 + 3x\partial_x + (1-m^2) + x^2]\ \{ x^{-1} f_m(x)\} = K\ x^2\ J_m(x) \qquad \text{(D.2.11)}$$

where

$$K_m \equiv 2j\ (\beta_d/\beta')\ C_{zm}\ . \qquad \text{(D.2.12)}$$

Ever eager, Maple computes the left side of (D.2.11),

```
g := fm(x)/x:
L := x*(x^2*diff(g,x,x)+3*x*diff(g,x) + ((1-m^2) + x^2)*g):
     simplify(%): expand(%): collect(%,fm(x));
```

$$(-m^2 + x^2)\,\mathrm{fm}(x) + \left(\frac{\partial^2}{\partial x^2}\mathrm{fm}(x)\right)x^2 + \left(\frac{\partial}{\partial x}\mathrm{fm}(x)\right)x$$

so that (D.2.11) becomes

$$[\ x^2\ \partial_x^2\ + x\ \partial_x + (x^2-m^2)]\ f_m(x) = K_m\ x^2\ J_m(x)\ . \qquad \text{(D.2.13)}$$

The left side of (D.2.13) is the normal Bessel operator [Spiegel 24.1] , but the equation is also *driven* by a power times a Bessel function. The solution to the equation is the homogeneous solution of the Bessel equation plus the particular solution which is the response to the driving function on the right hand side.

The homogeneous solution is the usual linear combination of $J_m(x)$ and $Y_m(x)$, but we must reject $Y_m(x)$ since it blows up at x=0 and thereby causes the field E_r to be singular, which it cannot be, smack in the middle of a wire.

The particular solution is not very obvious and required some hunting to find. It is this

$$f_m(x)^{particular} = (1/2)\ K_m\ [\ x\ J_{m+1}(x)\ ]\ . \qquad \text{(D.2.14)}$$

a Maple confirms, continuing the above code,

```
fm := (x) -> (1/2)*Km*x*BesselJ(m+1,x);
```

$$fm := x \to \frac{1}{2}\,Km\ x\ \mathrm{BesselJ}(m+1, x)$$

```
L: simplify(%);
```

$$x^2\ Km\ \mathrm{BesselJ}(m, x)$$

Therefore, we now have this full solution for $f_m(x)$

$$f_m(x) = f_m(x)^{particular} + f_m(x)^{homogeneous} = (1/2)\ K_m\ [\ x\ J_{m+1}(x)\ ] + a_m\ J_m(x)$$

and then from (D.2.9) the full solution E_r

$$E_r(r,m) = a_m\ x^{-1}\ J_m(x) + \frac{K_m}{2}\ J_{m+1}(x)\ . \qquad (D.2.15)$$

For each value of m, there are two as-yet undetermined constants, a_m and $(K_m/2)$. However, looking at (D.2.10), we see that, since $J_0(x) \approx 1$ for small x, we must have

$$a_0 = 0 \qquad (D.2.16)$$

to keep E_r finite at r = 0. We shall obtain expressions for a_m and $(K_m/2)$ below.

(b) The E_φ Solution

Start with div **E** = 0 as stated in (D.2.6),

$$\partial_r\ (r\ E_r) + \partial_\varphi E_\varphi + r\ \partial_z E_z = 0\ . \qquad \textit{(D.2.6)}$$

Repeat the steps (D.1.12) through (D.1.17) to convert the above equation to m space. The rules are that $\partial_\varphi \rightarrow jm$ and $\partial_z \rightarrow -j\beta_d$ and $E_i(r,\varphi,z,t) \rightarrow E_i(r,m)$, so that

$$\partial_r\ (r\ E_r(r,m)) - jm\ E_\varphi(r,m) + r\ (\ -j\beta_d\ )E_z(r,m) = 0\ .$$

Then insert (D.1.19) for $E_z(r,m)$ to get,

$$\partial_r\ (r\ E_r(r,m)) + jmE_\varphi(r,m) - j\beta_d\ r\ C_{zm}\ J_m(\beta' r) = 0$$

or

$$\partial_r\ (r\ E_r(r,m)) + jmE_\varphi(r,m) - (1/2)\ [2j(\beta_d/\beta')\ C_{zm}\ ]\ \beta' r\ J_m(\beta' r) = 0$$

or

$$\partial_r\ (r\ E_r(r,m)) + jmE_\varphi(r,m) - (1/2)\ K_m\ x\ J_m(x) = 0 \qquad \text{// } x = \beta' r \text{ as earlier}$$

so

$$jmE_\varphi(r,m) = (K_m/2)\ x\ J_m(x)\ - \partial_x\ (x\ E_r(r,m))\ . \qquad (D.2.17)$$

Next, insert (D.2.15) $E_r(r,m)$ so

$$jmE_\varphi(r,m) = (K_m/2)\ x\ J_m\ - \partial_x\ (x\ \{\ a_m\ x^{-1}\ J_m + \frac{K_m}{2}\ J_{m+1}\})$$

$$= (K_m/2)\ x\ J_m\ - \partial_x\ (a_m\ J_m + \frac{K_m}{2}\ x\ J_{m+1})$$

$$= -\ a_m\ J_m' + \frac{K_m}{2}\ [\ -\ x\ J_{m+1}' -\ J_{m+1} + xJ_m]\ . \qquad (D.2.18)$$

At this point we invoke the recurrence relations AS 10.6.2

10.6.2

$$\mathscr{C}_\nu'(z) = \mathscr{C}_{\nu-1}(z) - (\nu/z)\,\mathscr{C}_\nu(z),$$
$$\mathscr{C}_\nu'(z) = -\,\mathscr{C}_{\nu+1}(z) + (\nu/z)\,\mathscr{C}_\nu(z).$$

to write

$$J_{m+1}' = J_m - (m+1)x^{-1}J_{m+1} \qquad \text{first relation with } \nu = m+1$$
$$J_m' = -J_{m+1} + (m/x)J_m \qquad \text{second relation with } \nu = m \qquad \text{(D.2.19)}$$

Insert these into (D.2.18) to get

$$jmE_\varphi(r,m) = - a_m J_m' + \frac{K_m}{2} [- x J_{m+1}' - J_{m+1} + xJ_m]$$
$$= - a_m \{-J_{m+1} + (m/x)J_m \} + \frac{K_m}{2} [- x \{ J_m - (m+1)x^{-1}J_{m+1}\} - J_{m+1} + xJ_m]$$
$$= a_m J_{m+1} - a_m (m/x)J_m + \frac{K_m}{2} [- x J_m + (m+1) J_{m+1} - J_{m+1} + xJ_m]$$
$$= a_m J_{m+1} - a_m (m/x)J_m + \frac{K_m}{2} [m J_{m+1}]$$
$$= - a_m (m/x)J_m + (\frac{K_m}{2} m + a_m) J_{m+1}$$

Dividing by m then gives the final solution:

$$jE_\varphi(r,m) = - a_m x^{-1} J_m(x) + (\frac{K_m}{2} + \frac{a_m}{m}) J_{m+1}(x) \ . \qquad \text{(D.2.20)}$$

We have already noted that $a_0 = 0$. For m=0, (D.2.20) is invalid because we divided (D.2.18) by m. In fact, we really know nothing at all about the field $E_\varphi(r,m=0)$ from our solution method, because the E_φ fields were obtained from $\partial_\varphi E_\varphi$ in the div **E** = 0 equation (D.2.6). Since $E_\varphi(r,m=0)e^{j0\varphi}$ does not depend on φ, this simply gives 0, but does not tells us what $E_\varphi(r,m=0)$ might be. Presumably we could find it from the φ Helmholtz equation which we have so far ignored.

Conjecture: It seems likely that $E_\varphi(r,m=0) = 0$. In the m=0 partial wave where everything is symmetrical in azimuth, it is hard to imagine a constant (in φ) field E_φ circulating around the axis of the wire. Thus, we interpret the quantity $(\frac{K_m}{2} + \frac{a_m}{m})$ in (D.2.20) as being 0 when m=0.

(c) Application of the Boundary Conditions

We have two boundary conditions to impose:

$$E_r(r=a,m) = (j\omega/\sigma) N_m \qquad \text{(D.2.14)}$$

$$E_\varphi(r=a,m) = 0 \qquad \text{(D.2.15)}$$

The first is the radial charge pumping condition shown in (D.1.9) above, while the second is our ansatz assumed earlier that any z = constant circle on the wire surface be an equipotential.

These two boundary conditions serve to determine the two constants a_m and K_m, though a bit of algebra is required. The first step is to use ** and ** to write out the two boundary conditions as

$$a_m\ x_a^{-1}\ J_m(x_a) + \frac{K_m}{2}\ J_{m+1}(x_a)\ = (j\omega/\sigma)\ N_m \qquad (1)$$

$$-\ a_m\ x_a^{-1}\ J_m(x_a) + \left(\frac{K_m}{2}\ + \frac{a_m}{m}\right)\ J_{m+1}(x_a)\ = 0 \qquad (2)$$

Addition and subtraction of these equations gives two new equations,

$$\frac{K_m}{2}\ J_{m+1}(x_a)\ +\ \left(\frac{K_m}{2}\ + \frac{a_m}{m}\right)\ J_{m+1}(x_a)\ =\ (j\omega/\sigma)\ N_m \qquad (3)$$

$$2\ a_m\ x_a^{-1}\ J_m(x_a) - \frac{a_m}{m}\ J_{m+1}(x_a)\ = (j\omega/\sigma)\ N_m \qquad (4)$$

the second of which may be immediately solved for a_m

$$\frac{a_m}{m}\ = (j\omega/\sigma)\ N_m \frac{1}{J_{m-1}(x_a)} \quad .$$ which agrees with (D.2.16)

Using the recursion relation $2m\ x^{-1}\ J_m = [J_{m+1} + J_{m-1}]$, equation (2) may be solved to get

$$\left(\frac{K_m}{2}\ + \frac{a_m}{m}\right)\ =\ (j\omega/2\sigma)\ N_m \left[\frac{1}{J_{m+1}(x_a)}\ + \frac{1}{J_{m-1}(x_a)}\right]$$

Finally, subtracting ** from ** we find

$$\frac{K_m}{2}\ = (j\omega/2\sigma)\ N_m \left[\frac{1}{J_{m+1}(x_a)}\ - \frac{1}{J_{m-1}(x_a)}\right]$$

To summarize,

$$a_m\ =\ m(j\omega/\sigma)\ N_m \frac{1}{J_{m-1}(x_a)} \qquad m\geq 0$$

$$\frac{K_m}{2}\ =\ (j\omega/2\sigma)\ N_m \left[\frac{1}{J_{m+1}(x_a)}\ - \frac{1}{J_{m-1}(x_a)}\right] \qquad m\geq 0$$

$$\left(\frac{K_m}{2}\ + \frac{a_m}{m}\right) =\ (j\omega/2\sigma)\ N_m \left\{\frac{1}{J_{m+1}(x_a)}\ + \frac{1}{J_{m-1}(x_a)}\ \right\} \qquad m>0 \qquad (D.2.16)$$

ok to here

The last equation of the above set should not be used for m=0, due to the $\frac{a_m}{m} = \frac{0}{0}$ condition, but the other two are correct. Since $J_{-n} = (-1)^n J_n$, for m = 0 we have $J_{-1} = - J_1$ so K_0 becomes just

$$K_0 = 2\,(j\omega/\sigma)\,N_0\,\frac{1}{J_1(x_a)} \qquad \text{(D.2.17)}$$

We have now completely solved the problem. To review, here are the three fields gathered together in one spot. For E_z we use (D.2.8) to replace C_{zm} .

$$jE_\varphi(r,m) = - a_m\, x^{-1}\, J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right)\, J_{m+1}(x)$$

$$E_r(r,m) = a_m\, x^{-1}\, J_m(x) + \frac{K_m}{2}\, J_{m+1}(x)$$

$$E_z(r,m) = -j(\beta'/\beta_d)\,\frac{K_m}{2}\, J_m(x) \qquad \text{verify all 3} \qquad \text{(D.2.18)}$$

D.3 Statement of the Results

Although we have a complete solution to our problem, it is useful to normalize the results to some well defined absolute scale, rather than the factors N_m as is done above.

The first step in doing this is to compute the total current I in the wire,

$$I = \int_0^{2\pi} d\varphi \int_0^a r\, dr\, J_z(r,m)\, e^{jm\varphi} \qquad \text{(D.3.1)}$$

It is clear from (D.3.1) that only the m=0 partial wave can make any contribution, so we write

$$I = \int_0^a 2\pi r dr\, \sigma E_z(r,m=0) = 2\pi\sigma\,[-j(\beta'/\beta_d)\,]\,\frac{K_0}{2}\int_0^a r\, dr\, J_0(x)$$

$$= -2\pi\sigma j\,\frac{1}{\beta'\beta_d}\,\left\{\frac{K_0}{2}\right\}\int_0^{x_a} dx\, x\, J_0(x) = -2\pi\sigma j\,\frac{1}{\beta'\beta_d}\,[(j\omega/\sigma)\,N_0\,\{\frac{1}{J_1(x_a)}\}]\,x_a J_1(x_a)$$

$$= -2\pi\sigma j\,\frac{1}{a\beta_d}\,(j\omega/\sigma)\,N_0\,a^2 \qquad \text{(D.3.2)}$$

At this point it is convenient to introduce the DC resistance per unit length of our wire,

$$R_{dc} = \frac{1}{\sigma\pi a^2} \qquad \text{(D.3.3)}$$

along with a new symbol to indicate the relative surface charge moment,

$$\eta_m \equiv \frac{N_m}{N_0} \tag{D.3.4}$$

Using (D.3.3) and (D.3.4) in (D.3.2) gives

$$(j\omega/\sigma)\, N_m = (j/2)\,(a\beta_d)\,\eta_m\, I\, R_{dc} \tag{D.3.5}$$

We can use (D.3.5) to get our coefficients in (D.2.16) into a form that is then normalized to the total current I. Putting all the pieces together, we get:

E fields in a Round Wire, m > 0

$$jE_\varphi(r,m) = (j/4)\,(a\beta_d)\,\eta_m\, I\, R_{dc}\ \left[-\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \left\{ \frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m+1}(x)}{J_{m-1}(x_a)} \right\} \right]$$

$$E_r(r,m) = (j/4)\,(a\beta_d)\,\eta_m\, I\, R_{dc}\ \left[+\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \left\{ \frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m+1}(x)}{J_{m-1}(x_a)} \right\} \right]$$

$$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\ \left[\frac{x_a J_m(x)}{J_{m+1}(x_a)} - \frac{x_a J_m(x)}{J_{m-1}(x_a)} \right] \tag{D.3.6}$$

E fields in a Round Wire, m = 0

$$jE_\varphi(r,m) = 0$$

$$E_r(r,m) = (j/2)\,(a\beta_d)\, I\, R_{dc}\ \left[\frac{J_1(x)}{J_1(x_a)} \right]$$

$$E_z(r,m) = (1/2)\, I\, R_{dc}\ \left[\frac{x_a\, J_0(x)}{J_1(x_a)} \right] \tag{D.3.7}$$

Comments about the solution:

(1) Notice that the components E_φ and E_r are smaller than E_z by factor ($\beta_d a$). This is the dimensionless smallness parameter which defines the "transmission line limit", see Chapter 4.

(2) If there exist moments N_m of the charge distribution on the wire with m > 1, then the corresponding $\eta_m \neq 0$, and it is clear that E_z and hence J_z is *non-uniform* across the surface of the wire. The non-uniformity is not "small" but has the full strength of η_m. Of course we only expect to get significant moments of charge density n(φ) when conductors are "fat and close". It seems likely in this case that the largest contribution will come from m=1.

(3) Because E_z is non-uniform, we conclude that the *surface impedance* $Z_s = E_z/I$ will also be non-uniform around the boundary of the wire. In fact, here is our explicit formula for the surface impedance of a round wire. At r=a, it is a function only of azimuth angle φ :

$$Z_s(\varphi) = (1/2)R_{dc}\ \{\ [\ \frac{x_a\ J_0(x_a)}{J_1(x_a)}\ ]\ +\ \sum_{m=1}^{\infty}\ [\ \frac{x_a J_m(x_a)}{J_{m+1}(x_a)}\ -\ \frac{x_a J_m(x_a)}{J_{m-1}(x_a)}\ ]\ \mathrm{Re}[\eta_m e^{jm\varphi}]\ \ \}\qquad (D.3.8)$$

D.4 The Low Frequency Limit

If we assume that ω is small enough that $x_a = \beta' a << 1$, we can greatly simplify the above results. Another way to state this limit is that $(a/\delta) << 1$, which means the skin depth is much larger than the radius of the wire.

All limits come from the leading term of $J_n(x)$ which is

$$J_n(x) \approx x^n/(2^n m!) \qquad \text{must be} \qquad J_n(x) \approx x^n/(2^n n!) \qquad (D.4.1)$$

In the formulas for E_φ and E_r in the box (D.3.6), the leading term is the first term, the other terms may be neglected. We find that,

$$\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)}\ \approx (r/a)^{m-1} \qquad (D.4.2)$$

and the first factor in E_z is its leading term,

$$\frac{x_a J_m(x)}{J_{m+1}(x_a)}\ \approx 2(m+1)\ (r/a)^m \qquad (D.4.3)$$

For the m=0 fields we notes these facts,

$$[\ \frac{J_1(x)}{J_1(x_a)}\ ] \approx (r/a) \qquad [\ \frac{x_a\ J_0(x)}{J_1(x_a)}\ ] \approx 2 \qquad (D.4.4)$$

So here is a complete summary of the low frequency results:

E fields in a Round Wire, m > 0, Low Frequency Limit

$$jE_\varphi(r,m) = (j/4)\,(a\beta_d)\,\eta_m\,I\,R_{dc}\,[\,-(r/a)^{m-1}\,]$$

$$E_r(r,m) = (j/4)\,(a\beta_d)\,\eta_m\,I\,R_{dc}\,[\,+(r/a)^{m-1}\,]$$

$$E_z(r,m) = (1/2)\,\eta_m\,I\,R_{dc}\,[\,(m+1)\,(r/a)^m\,] \qquad \text{(D.4.5)}$$

E fields in a Round Wire, m = 0, Low Frequency Limit

$$jE_\varphi(r,0) = 0$$

$$E_r(r,0) = (j/2)\,(a\beta_d)\,I\,R_{dc}\,(r/a)$$

$$E_z(r,0) = I\,R_{dc} \qquad \text{(D.4.6)}$$

In this very last equation, we recover the fact that a piece of wire really does act like a resistor at low frequencies. However, there is still a small linear radial field which serves to feed the symmetric surface charge in the m=0 partial wave. In fact, if you integrate the radial current over the surface of a long piece of wire of length λ/ 2, taking into account the z dependence, you get 2I.

See Section 3.7, Figure 2 for a drawing of the current we have just computed.

D.5 What about φ, A and B ?

Inside the conductor, we know that φ satisfies (D.1.1), and we know that φ is constant at r=a. This means that φ is similar to our E_z solution in the m=0 partial wave, and vanishes for all higher partial waves. Thus we write

$$\varphi(r,m) = \delta_{m,0}\;\varphi_0\,J_0(x)/J_0(x_a) \qquad x = \beta' r \qquad \text{(D.5.1)}$$

where φ_0 is the value of the potential on the surface at r=a. Since we know **E**(r,m) from Section 3 (6), we can solve for the vector potential **A** as follows:

$$-j\omega\mathbf{A} = \mathbf{E} + \text{grad}\,\varphi \qquad \text{(D.5.2)}$$

Converted to partial waves, this says

$$-j\omega A_r(r,m) = E_r(r,m) + \delta_{m,0}\;\partial_r\varphi(r,0)$$

$$-j\omega A_\varphi(r,m) = E_\varphi(r,m) - (1/r)\,jm\;\delta_{m,0}\;\varphi(r,0) = E_\varphi(r,m)$$

$$-j\omega A_z(r,m) = E_z(r,m) - j\beta_d\;\delta_{m,0}\;\varphi(r,0) \qquad \text{(D.5.3)}$$

Thus, for m≠0 we have -jω**A** = **E.** For m = 0 there are extra pieces as shown for $A_r(r,0)$ and $A_z(r,0)$.

Since **A**(r,m) is known, **A**(r,φ,z) is also known, and then so too is **B** = curl **A**. This curl can also be performed in each partial wave if desired by replacing $\partial_\varphi = -jm$ and $\partial_z = -j\beta_d$ as usual.

Since -jω**A** = **E** in the higher partial waves, we may conclude that the transverse components of **A** are small compared to the longitudinal component, just as is the case for **E**, see (D.3.6). For m=0 we know that $A_\varphi = E_\varphi = 0$, but A_r is a combination of E_r and $\partial_r \varphi$ which is *not* small compared to A_z, due to the $\partial_r\varphi$ contribution.

Thus, we arrive at the conclusion that, although A_r is negligible in the dielectric, it cannot be ignored inside the conductor. This explains, incidentally, the problem one encounters with the gauge condition (1.5.5) div**A** = $-j(\beta^2/\omega)\varphi$. Since φ is continuous at the boundary, and β^2 takes a jump of many orders of magnitude (from β_d to β'), something on the left side must change violently. But A_z is also continuous. It is the m=0 A_r inside the conductor that takes up the slack. In fact, taking the difference inside minus outside we conclude that

$$\partial_r A_r\,(r=a-\varepsilon) = -j(\beta'^2/\omega)\varphi_0 \qquad \text{(D.5.4)}$$

from which we can determine φ_0 The potential A_r is discontinuous at r=a.