Saving old Section D.11 PhL 6.24.14

D.11 Low frequency limit of the round wire E fields

Recall from (D.2.33) that,

Summary of the E field solutions : $R_{dc} = \frac{1}{\sigma\pi a^2}$ $\beta'^2 = \beta^2 - \beta_d^2$ *(D.2.33)*

$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta')\, f_m$ $f_m = [\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}]$ $x = \beta' r$

$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, g_m$ $g_m = [\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}]$ $x_a = \beta' a$

$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, h_m$ $h_m = [\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}]$

Our goal here is to evaluate f_m, g_m and h_m in the low ω limit. For low frequency we continue to use

$$\beta = e^{j3\pi/4}\,(\sqrt{2}/\delta) = (j-1)/\delta = e^{j3\pi/4}\sqrt{\sigma\omega\mu} \qquad (2.2.20) \qquad (2.2.21)$$

(a) The effect of including loss on the parameter β'

If for the wave's wavenumber β_d one uses the usual lossless $\beta_d = (\omega/v_d)$ one finds,

$$\frac{\beta_d}{\beta} = \frac{(\omega/v_d)}{e^{j3\pi/4}\sqrt{\sigma\omega\mu}} = \text{constant} * \sqrt{\omega} \qquad \text{// no loss} \qquad (D.11.1)$$

which results in $(\beta_d/\beta) \rightarrow 0$ for the low ω limit. Since $\beta'^2 = \beta^2 - \beta_d^2$ this again gives $\beta' = \beta$ as in Section D.10. But now this β is very small at low ω, instead of very large as in the Section D.10. This is the correct limit, but we want to make sure it is still the correct limit if we include cable loss.

Including in β_d a model for the unavoidable loss, we find at low ω (see section (b) below) that

$$\beta_d = (\omega - j\omega_c)/v_d \qquad \text{where} \quad \omega_c = (v_d^2/2)\,(R_{dc}C) \qquad (D.11.2)$$

$$E(z) = E(0)\, e^{-j\beta_d z} = e^{-jz(\omega - j\omega_c)/v_d} = e^{-j(\omega/v_d)z}\, e^{-(\omega_c/v_d)z}$$

where R_{dc} is the DC resistance per unit length of the transmission line (both conductors) and C is the capacitance per unit length. Reconsider now the low ω situation:

$$\frac{\beta_d}{\beta} = \frac{(\omega - j\omega_c)/v_d}{e^{j3\pi/4}\sqrt{\sigma\omega\mu}} \rightarrow \text{constant}/\sqrt{\omega}\ . \qquad \text{// loss} \qquad (D.11.3)$$

Now $(\beta_d/\beta) \rightarrow \infty$ for the low ω limit, just the reverse of the lossless situation. Since $\beta'^2 = \beta^2 - \beta_d^2$ one finds from (D.11.2) that, for low ω,

$$\beta'^2 \approx -\beta_d^2 = -(j\omega_c)^2/v_d^2 = (\omega_c/v_d)^2$$

so

$$\beta' \approx (\omega_c/v_d) . \qquad \text{(D.11.4)}$$

But since ω_c is usually fairly small and v_d is fairly large, it turns out that $a\beta' << 1$ and we are then still interested in the small β' limit of our E field coefficients like f_m, as shown in an example below.

Alternatively one can write,

$$\beta_d = \beta_{d0} - j\alpha$$
$$\beta_{d0} = (\omega/v_d)$$
$$\alpha = (\omega_c/v_d) = (v_d/2)(R_{dc}C) \qquad \dim(\alpha) = (m/sec)^* \, sec/m^2 = m^{-1} \qquad \text{(D.11.5)}$$

where α is the amplitude loss factor for a propagating wave

$$e^{-j\beta_d z} = e^{-j\beta_{d0} z} e^{-\alpha z}$$

$$\mathrm{Re}(e^{-j\beta_d z}) = e^{-\alpha z} \cos(\beta_{d0} z) = e^{-(\omega_c/v_d) z} \cos[(\omega/v_d)z] . \qquad \text{(D.11.6)}$$

(b) Derivation of the simple loss model

Start with this expression for power P in a wave,

$$P = IV = U v_d = [(1/2)L I^2 + (1/2)CV^2] v_d . \qquad \dim(Uv_d) = J/m^*m/sec = watts \qquad \text{(D.11.7)}$$

Then

$$dP/dz = [L I \, dI/dz + C V \, dV/dz] v_d$$

$$= [L I (-\mathbf{y} V) + C V (-\mathbf{z} I)] v_d \qquad \text{// using transmission line equations (4.11.14b)}$$

$$= -IV [L\mathbf{y} + C\mathbf{z}] v_d = -P[L(G + j\omega C) + C(R + j\omega L)] v_d$$

$$= -2\alpha P \qquad \text{(D.11.8)}$$

where

$$\alpha = [L(G + j\omega C) + C(R + j\omega L)] (v_d/2) . \qquad \text{(D.11.9)}$$

So then

$P(z) = e^{-2\alpha z} P(0)$

$|\mathbf{E}(z)| = e^{-\alpha z} |\mathbf{E}(0)|$. // electric field and all other non-power quantities (D.11.10)

Ignoring G and assuming low ω, we obtain the result quoted above,

$$\alpha = [\, L(G + j\omega C) + C(R + j\omega L)] \, (v_d/2) \approx (LG+RC) \, (v_d/2) \approx (R_{dc}C) \, (v_d/2)$$

$$\omega_c = v_d\alpha = (R_{dc}C) \, (v_d^2/2) \qquad f_c = (R_{dc}C) \, (v_d^2/4\pi) \qquad \text{(D.11.11)}$$

The main point is that in this simple model α and ω_c are constants, independent of ω.

Example 1: Consider Belden 8281 coaxial cable :

Electrical Characteristics (Overall)

Nom. Characteristic Impedance:

Impedance (Ohm)
75

Nom. Inductance:

Inductance (µH/m)
0.429811

Nom. Capacitance Conductor to Shield:

Capacitance (pF/m)
68.901

Nominal Velocity of Propagation:

VP (%)
66

Nominal Delay:

Delay (ns/m)
5.05274

Nom. Conductor DC Resistance:

DCR @ 20°C (Ohm/km)
32.4819

Nominal Outer Shield DC Resistance:

DCR @ 20°C (Ohm/km)
3.6091

Nom. Attenuation:

Freq. (MHz)	Attenuation (dB/100m)
1	0.9843
3.6	1.6405
10.0	2.6248
71.5	6.8901
135	9.843
270	14.1083
360	16.7331
540	20.6703
720	24.2794
750	24.9356
1000	30.1852

For this standard 75Ω cable one finds from (D.11.2) and $f_c = \omega_c/2\pi$,

$R_{dc} = (32.5 + 3.6)\ \Omega/\text{km} = 36.1 \times 10^{-3}$ ohms/m

$C = 69$ pF/m $= 69 \times 10^{-12}$ farad/m

$v_d = 0.66c = 0.66 \times 3 \times 10^8$ m/sec $= 2 \times 10^8$ m/sec

```
Rdc := 36.1e-3;
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$$Rdc := .0361$$

```
C := 69e-12;
```

$$C := .69\ 10^{-10}$$

```
vd := 0.66 * 3e8;
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$$vd := .198\ 10^{9}$$

```
fc := Rdc*C*vd^2/(4*evalf(Pi));
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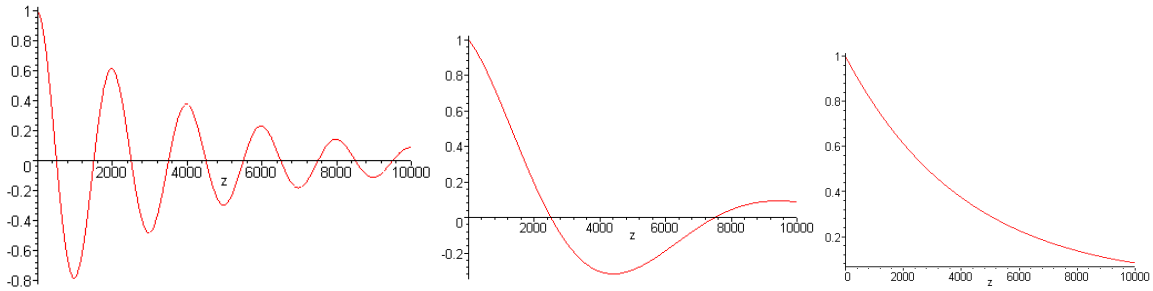
$$fc := 7770.998213$$

Thus f_c = 7770 Hz and $\omega_c = 2\pi f_c$ = 48,827. So any line quantity like V(z) has this behavior,

$$V(z) = V(0)\, e^{-(\omega_c/v_d)z} \cos[(\omega/v_d)z] = V(0)\, e^{-(2\pi f_c/v_d)z} \cos[2\pi(f/v_d)z] \qquad \text{(D.11.12)}$$

which we can plot for V(z) over 10 km of this Belden cable for several values of f:



f = 100 KHz f = 10 KHz f = 1 KHz Fig D.9

In all cases, some signal arrives at the end (ignoring noise), but in the right two graphs one would say the cable was quite "lossy".

For this example, we can compute β' for low ω using (D.11.4),

$$\beta' = (\omega_c/v_d) = 48{,}827 / 2{*}10^8 = .00024 = 2.4 \times 10^{-4}\ m^{-1}$$

$$a = 394\mu = 394 \times 10^{-6}\ m = 0.394 \times 10^{-3}\ m \qquad \text{// wire radius}$$

$$x_a = \beta' a = 2.4 \times 10^{-4} * 0.394 \times 10^{-3} = 0.95 \times 10^{-7} \qquad \text{(D.11.13)}$$

so at low ω it is appropriate to assume that $x_a = \beta' a$ and $x = \beta' r$ are both << 1.

Example 2: At the end of Section 4.5 we considered a power transmission line with two 1" diameter conductors separated by 1 meter. It was found that K = 17.5 and that R = .02Ω per thousand feet for each conductor. Thus $R_{dc} \approx 1.2 \times 10^{-4}$ ohms/m for both conductors and $C = 4\pi\varepsilon_0/K = 6.35 \times 10^{-12}$ F/m. We then find that $\omega_c = (c^2/2)(R_{dc}C) = 34.3\ sec^{-1}$ and $f_c = 5.5$ Hz. Such power lines are normally operated at 50 or 60Hz which is about $10f_c$ which corresponds roughly to the left graph in Fig D.9 above.

Comment: Since the transmission line equations were used in deriving the above loss model, and since the transmission line equations are only valid in the strong or extreme skin effect regimes, the loss model is not valid for "very low frequencies" which we might take to mean $f < f_c$. In other words, the loss model is really only valid if the loss is small, and at sufficiently low ω the loss is *not* small and the model does not apply. In particular, the model does not apply at ω= 0 or very close to ω = 0.

(c) Low frequency evaluation of f_m, g_m and h_m and the E fields

In the following we consider only $m \geq 0$ since we know from (D.10.2) that $f_{-m} = f_m$, $g_{-m} = g_m$, $h_{-m} = h_m$.

The small x limit for $J_m(x)$ is given by NIST 10.7.3,

$$J_n(x) = (x/2)^n / n! \quad . \qquad \text{for } n = 0,1,2,..... \qquad \text{(D.11.14)}$$

Since J_{m-1} appears in our coefficient expressions and since m = 0 is encountered, we have to deal with m = 0 as a special case since the above limit is not valid for n = -1. To this end we use NIST 10.2.2 which is valid for integer n,

$$J_{-n}(x) = (-1)^n J_n(x) \approx (-1)^n (x/2)^n / n! \qquad \text{(D.11.15)}$$

so that $J_{-1}(x) = - J_1(x) \approx - (x/2)$. Our small-x forms of interest are then

$$\begin{aligned} &J_n(x) = (x/2)^n / n! && \text{for } n = 0,1,2,..... \\ &J_{-1}(x) = - (x/2) && \text{for } n = -1 \quad . \end{aligned} \qquad \text{(D.11.16)}$$

We now examine the small x limits of f_m, g_m, and h_m .

First f_m for m > 0, and then for m = 0:

$$f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] = \left[\frac{(x/2)^m/m!}{(x_a/2)^{m+1}/(m+1)!} - \frac{(x/2)^m/m!}{(x_a/2)^{m-1}/(m-1)!}\right]$$

$$= [\,(m+1)\,(x/x_a)^m\,(2/x_a) - (1/m)\,(x/x_a)^m(x_a/2)\,]$$

$$= (x/x_a)^m\,[\,(m+1)\,(2/x_a) - (1/m)\,(x_a/2)\,]$$

$$\approx (x/x_a)^m\,(m+1)\,(2/x_a) \qquad \text{// as } x_a \rightarrow 0$$

$$f_0 = \left[\frac{J_0(x)}{J_1(x_a)} - \frac{J_0(x)}{J_{-1}(x_a)}\right] = \left[\frac{J_0(x)}{J_1(x_a)} + \frac{J_0(x)}{J_1(x_a)}\right] = 2\,\frac{J_0(x)}{J_1(x_a)} = 2\,\frac{1}{(x_a/2)} = 4/x_a$$

First g_m for m > 0, and then for m = 0:

$$g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)} \right] = \left[\frac{(x/2)^{m+1}/(m+1)!}{(x_a/2)^{m+1}/(m+1)!} + \frac{(x/2)^{m-1}/(m-1)!}{(x_a/2)^{m-1}/(m-1)!} \right]$$

$$= (x/x_a)^{m+1} + (x/x_a)^{m-1}$$

$$g_0 = \left[\frac{J_1(x)}{J_1(x_a)} + \frac{J_{-1}(x)}{J_{-1}(x_a)} \right] = \left[\frac{J_1(x)}{J_1(x_a)} + \frac{J_1(x)}{J_1(x_a)} \right] = 2 \frac{J_1(x)}{J_1(x_a)} = 2\,(x/x_a)$$

Results for h_m are then obvious since there is only a sign change between the terms in g_m,

$$h_m = (x/x_a)^{m+1} - (x/x_a)^{m-1} \qquad h_0 = 0$$

The results are then:

Small ω limit of the E field solutions : $R_{dc} = \dfrac{1}{\sigma \pi a^2}$ (D.11.17)

$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta')\, f_m \qquad f_m = (r/a)^m\,(m+1)\,(2/\beta' a) \qquad f_0 = 4/(a\beta')$

$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, g_m \qquad g_m = (r/a)^{m+1} + (r/a)^{m-1} \qquad g_0 = 2\,(r/a)$

$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta_d)\, h_m \qquad h_m = (r/a)^{m+1} - (r/a)^{m-1} \qquad h_0 = 0$

Recall now these field expansions for the case that n(θ) is an even function of θ,

$$E_z(r,\theta) = (1/4)\, I\, R_{dc}\, (a\beta') \left[f_0 + 2 \sum_{m=1}^{\infty} f_m\, \eta_m \cos(m\theta) \right]$$

$$E_r(r,\theta) = (j/4)\, I\, R_{dc}\, (a\beta_d) \left[g_0 + 2 \sum_{m=1}^{\infty} g_m\, \eta_m \cos(m\theta) \right]$$

$$E_\theta(r,\theta) = (1/4)\, I\, R_{dc}\, (a\beta_d) \left[h_0 + 2 \sum_{m=1}^{\infty} h_m\, \eta_m \cos(m\theta) \right] . \qquad \textit{(D.10.4a)}$$

For low ω we insert the expressions above to get

$$E_z(r,\theta) = (1/4)\, I\, R_{dc} \left[4 + 4 \sum_{m=1}^{\infty} (r/a)^m\, (m+1)\, \eta_m \cos(m\theta) \right]$$

$$E_r(r,\theta) = (j/4)\, I\, R_{dc}\, (a\beta_d) \left[2\,(r/a) + 2 \sum_{m=1}^{\infty} \left[(r/a)^{m+1} + (r/a)^{m-1} \right] \eta_m \cos(m\theta) \right]$$

$$E_\theta(r,\theta) = (1/4)\, I\, R_{dc}\, (a\beta_d) \left[2 \sum_{m=1}^{\infty} \left[(r/a)^{m+1} - (r/a)^{m-1} \right] \eta_m \cos(m\theta) \right] . \qquad \text{(D.11.18)}$$

Observations on the E fields for small ω

1. The fields E_r and E_θ are smaller than the field E_z by factor $(a\beta_d)$.

2. The m = 0 term in E_z is just $E_z = IR_{dc}$ which says $J_z = \sigma\, IR_{dc} = \sigma\, I(1/\sigma\pi a^2) = I/(\pi a^2)$. This is the current one would expect in a wire carrying DC current I.

3. Assuming a non-uniform n(θ) charge density on the wire surface, the η_m moments are non-zero and one concludes that $J_z(r,\theta)$ is asymmetric even as $\omega \to 0$. This conclusion is incorrect, and the reason it is incorrect is explained in Chapter 6.5 (e). One item to note is that

$$I = V/Z_0$$

and according to (4.11.16), where G = 0 since we assumed a non-conducting dielectric,

$$Z_0 \equiv V(z)/i(z) = \sqrt{\frac{\mathbf{z}}{\mathbf{y}}} = \sqrt{\frac{R + j\omega L}{j\omega C}} \ . \qquad (4.11.16)$$

Thus in the limit $\omega \to 0$ we get $Z_0 \to \infty$. For a finite length transmission line, as one lowers ω, one must increase the size of the termination Z_0 to maintain a properly terminated line. Thus, our situation here does not apply to taking the low frequency limit of a transmission line terminated by a fixed 75Ω or 8Ω resistor. See Section 6.5 (e) for more on this subject. The major point here is that the entire model is not valid for low ω and we cannot use it to study the limit $\omega \to 0$.