Old Sections D.4, D.5 and D.6 (retired on 9/21/14) 9.21.14

D.4 Computation of the B fields in the round wire

The **B** field components may be computed from the Maxwell curl **E** equation (1.1.2)

$$-\partial_t \mathbf{B} = \text{curl } \mathbf{E} \ . \qquad \text{Maxwell curl E equation} \qquad \textit{(1.1.2)} \qquad \text{(D.4.1)}$$

In cylindrical coordinates one has from (D.1.14),

$$\text{curl } \mathbf{E} = \hat{\mathbf{r}} \left[r^{-1}\partial_\theta E_z - \partial_z E_\theta \right] + \hat{\boldsymbol{\theta}} \left[\partial_z E_r - \partial_r E_z\right] + \hat{\mathbf{z}} \left[r^{-1}\partial_r (rE_\theta) - r^{-1}\partial_\theta E_r \right] \qquad \text{(D.4.2)}$$

where the fields are of the traveling wave form shown in (D.1.1) which we assume also for the **B** field. Thus, combining (D.1.1) with (D.1.3a), one has

$$\mathbf{E}(r,\theta,z,t) = e^{j(\omega t - kz)}\, \mathbf{E}(r,\theta) = e^{j(\omega t - kz)} \sum_{m=-\infty}^{\infty} \mathbf{E}(r,m)\, e^{jm\theta} \qquad \text{(D.4.3)}$$

$$\mathbf{B}(r,\theta,z,t) = e^{j(\omega t - kz)}\, \mathbf{B}(r,\theta) = e^{j(\omega t - kz)} \sum_{m=-\infty}^{\infty} \mathbf{B}(r,m)\, e^{jm\theta} \ . \qquad \text{(D.4.4)}$$

Inserting the three cylindrical components of the **E** expansion (D.4.3) into (D.4.2), one finds that these replacements may be made,

$$\partial_t \rightarrow +j\omega \qquad \partial_z \rightarrow -jk \qquad \partial_\theta \rightarrow +jm \ . \qquad \text{(D.4.5)}$$

Similarly, inserting the **B** expansion (D.4.4) into $-\partial_t \mathbf{B}$ one may replace $\partial_t \rightarrow +j\omega$. After doing this, both sides of (D.4.1) are expansions having the general form of (D.4.3) and one may then equate terms in the m sum [completeness of the $e^{jm\theta}$ on $(-\pi.\pi)$] to find that

$$-j\omega \mathbf{B}(r,m) = \hat{\mathbf{r}} \left[r^{-1} jmE_z + jkE_\theta \right] + \hat{\boldsymbol{\theta}} \left[-jkE_r - \partial_r E_z \right] + \hat{\mathbf{z}} \left[r^{-1}\partial_r (rE_\theta) - r^{-1} jmE_r \right] \qquad \text{(D.4.6)}$$

and this then gives the three components of the **B** field

$$B_r(r,m) = (j/\omega)\, [\text{curl } E]_r = (j/\omega) \left[r^{-1} jmE_z + jkE_\theta \right]$$

$$B_\theta(r,m) = (j/\omega)\, [\text{curl } E]_\theta = (j/\omega) \left[-jkE_r - \partial_r E_z \right]$$

$$B_z(r,m) = (j/\omega)\, [\text{curl } E]_z = (j/\omega) \left[r^{-1}\partial_r (rE_\theta) - r^{-1} jmE_r \right] \ . \qquad \text{(D.4.7)}$$

It is now a mechanical task to insert our E field components, and such tasks are grist for Maple's mill. We use the E component forms summary box (D.2.21) which have the a_m and K_m constants not yet specified.

```
restart;
alias(I=I,j=sqrt(-1),J=BesselJ):
```
Enter the three cyl coords curlE expressions from (D.4.6)
```
curlE_r := (1/r)*j*m*Ez + j*k*Ephi:
curlE_phi := -j*k*Er - Diff(Ez,r):
curlE_z := (1/r)*Diff(r*Ephi,r) - (1/r)*j*m*Er:
```
Enter the E fields from (D.2.21)
```
Ez := -j*(bp/k)*(Km/2)*J(m,x):
Er := am*(1/x)*J(m,x) + (Km/2)*J(m+1,x):
Ephi := (-j)*(-am*(1/x)*J(m,x) + (Km/2+am/m)*J(m+1,x)):
x := bp*r:  set x
```
Compute the three B field components from the Maxwell curl E equation
```
Br:=(j/w)*simplify(curlE_r):
Bphi := (j/w)*simplify(value(curlE_phi)):
Bz := (j/w)*simplify(value(curlE_z)):
```

The alias line "unaliases" I, sets $j = \sqrt{-1}$ in place of the default I, and allows simple reference to the Bessel functions of interest. Diff(Ez,r) represents $\partial_r E_z$, but in an "inert" form which is not executed until later after E_z has been specified. The resulting B field expressions are somewhat ugly but can be cleaned up using a few more Maple manipulations. Having seen the results, we extract certain factors as shown in the following commands,

```
collect(expand((w/bp)*Bz),J);
```
$$\left(\frac{1}{2}Km+\frac{am}{m}\right)\mathrm{J}(m,bp\,r)$$
```
collect(expand((w/(j*bp))*Br),J);
```
$$\left(\frac{1}{2}\frac{m\,Km}{r\,k}-\frac{k\,am}{bp^2\,r}\right)\mathrm{J}(m,bp\,r)+\left(\frac{1}{2}\frac{k\,Km}{bp}+\frac{k\,am}{bp\,m}\right)\mathrm{J}(m+1,bp\,r)$$
```
collect(expand((w/bp)*Bphi),J);
```
$$\left(\frac{k\,am}{bp^2\,r}-\frac{1}{2}\frac{m\,Km}{r\,k}\right)\mathrm{J}(m,bp\,r)+\left(\frac{1}{2}\frac{k\,Km}{bp}+\frac{1}{2}\frac{bp\,Km}{k}\right)\mathrm{J}(m+1,bp\,r)$$

which we then translate back into our normal notation,

$$(\omega/\beta')B_z(r,m) = \left(\frac{K_m}{2}+\frac{a_m}{m}\right)J_m(x)$$

$$(\omega/j\beta')B_r(r,m) = +\left(m\frac{K_m}{2}\frac{1}{rk}-\frac{k}{r\beta'^2}a_m\right)J_m(x)+\frac{k}{\beta'}\left(\frac{K_m}{2}+\frac{a_m}{m}\right)J_{m+1}(x)$$

$$(\omega/\beta')B_\theta(r,m) = -\left(m\frac{K_m}{2}\frac{1}{rk}-\frac{k}{r\beta'^2}a_m\right)J_m(x)+\frac{K_m}{2}\left(\frac{k}{\beta'}+\frac{\beta'}{k}\right)J_{m+1}(x)\,. \qquad \text{(D.4.8)}$$

The last two equations contain the same factor which can be written as (recall $x = r\beta'$)

$$\left(m\frac{K_m}{2}\frac{1}{rk}-\frac{k}{r\beta'^2}a_m\right)=\left(m\frac{K_m}{2}\frac{\beta'}{r\beta' k}-\frac{k}{r\beta'^2}a_m\right)=\left(m\frac{K_m}{2}\frac{\beta'}{k}-\frac{k}{\beta'}a_m\right)(1/x)$$

The three equations for the exact **B** field components in the round wire are then shown in the summary box below which includes the earlier **E** field results as well:

Summary of E and B fields inside a round wire (D.4.9)

$$E_z(r,m) = -\,j\,(\beta'/k)\,\frac{K_m}{2}\,J_m(x) \qquad x = \beta' r \qquad \beta'^2 = \beta^2 - k^2$$

$$E_r(r,m) = a_m\,x^{-1}\,J_m(x) + \frac{K_m}{2}\,J_{m+1}(x)\ .$$

$$jE_\theta(r,m) = -\,a_m\,x^{-1}\,J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x) \qquad \textit{(D.2.21)}$$

$$B_z(r,m) = (\beta'/\omega)\left(\frac{K_m}{2} + \frac{a_m}{m}\right)J_m(x)$$

$$B_r(r,m) = j(\beta'/\omega)\left\{ + \left(m\frac{K_m}{2}\frac{\beta'}{k} - \frac{k}{\beta'}a_m\right) x^{-1}J_m(x) + \frac{k}{\beta'}\left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x) \right\}$$

$$B_\theta(r,m) = (\beta'/\omega)\left\{ - \left(m\frac{K_m}{2}\frac{\beta'}{k} - \frac{k}{\beta'}a_m\right) x^{-1}J_m(x) + \frac{K_m}{2}\left(\frac{k}{\beta'} + \frac{\beta'}{k}\right) J_{m+1}(x) \right\} \quad , \qquad \textit{(D.4.8)}$$

where the constants are given in (D.2.28), which we rewrite using (D.2.32),

$$a_m = (j/4)\,(ak)\,\eta_m\,I\,R_{dc} \;*\; 2m\,\frac{1}{J_{m-1}(x_a)}$$

$$\frac{K_m}{2} = (j/4)\,(ak)\,\eta_m\,I\,R_{dc} \;*\; \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right]$$

$$\left(\frac{K_m}{2} + \frac{a_m}{m}\right) = (j/4)\,(ak)\,\eta_m\,I\,R_{dc} \;*\; \left[\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\right] \ .$$

The three constant quantities at the end of the above summary box are roughly the same size in terms of scale. Using this fact, and the fact that for a low-loss line $|\beta'| >> |k| = |\beta_d|$ (so $\beta = \sqrt{\beta'^2 + k^2} \approx \beta'$) we can simplify (D.4.9) to read,

$$B_z(r,m) = (\beta/\omega)\left(\frac{K_m}{2} + \frac{a_m}{m}\right)J_m(x) \qquad \beta' \approx \beta, \quad x = \beta r$$

$$B_r(r,m) = j(\beta/\omega)\left\{ + \left(m\frac{K_m}{2}\frac{\beta}{k}\right) x^{-1}J_m(x) \right\} \qquad // \ m \neq 0$$

$$B_\theta(r,m) = (\beta/\omega)\left\{ - \left(m\frac{K_m}{2}\frac{\beta}{k}\right) x^{-1}J_m(x) + \frac{K_m}{2}\left(\frac{\beta}{k}\right) J_{m+1}(x) \right\} . \qquad \text{(D.4.10)}$$

The last line of (D.4.10) can be further simplified,

$$B_\theta(r,m) = (\beta/\omega)\{ - (m\frac{K_m}{2}\frac{\beta}{k}) x^{-1}J_m(x) + \frac{K_m}{2}(\frac{\beta}{k}) J_{m+1}(x) \}$$

$$= (\beta/\omega)\frac{K_m}{2}\frac{\beta}{k}\frac{1}{2}\{ - 2mx^{-1}J_m(x) + 2J_{m+1}(x) \}$$

$$= (\beta/\omega)\frac{K_m}{2}\frac{\beta}{k}\frac{1}{2}\{ - J_{m+1}(x) - J_{m-1}(x) + 2J_{m+1}(x) \} \qquad \text{// Spiegel 24.17 identity}$$

$$= (\beta/\omega)\frac{K_m}{2}\frac{\beta}{k}\frac{1}{2}\{ J_{m+1}(x) - J_{m-1}(x) \} \ . \qquad \text{(D.4.11)}$$

Therefore, in the limit $|\beta'| >> |k| = |\beta_d|$ equations (D.4.10) become

$$B_z(r,m) = (\beta/\omega)(\frac{K_m}{2} + \frac{a_m}{m})J_m(x) \qquad \beta' \approx \beta$$

$$B_r(r,m) = j(\beta/\omega)\frac{\beta}{\beta_d}\{ m\frac{K_m}{2} x^{-1}J_m(x) \} \qquad // \ m \neq 0$$

$$B_\theta(r,m) = (\beta/\omega)\frac{\beta}{\beta_d}\frac{K_m}{2}\frac{1}{2}[J_{m+1}(x) - J_{m-1}(x)] \ . \qquad \text{(D.4.12)}$$

For m>0, $|B_r|$ and $|B_\theta|$ are larger than $|B_z|$ by the large factor $|\beta/\beta_d|$.
For m = 0, $B_r \approx 0$ and $|B_\theta| >> |B_z|$. It is this large B_θ field which appears in Chapter 2 as B_θ.

D.5 Verification that the E and B fields satisfy the Maxwell equations

The Maple program discussed above goes on to verify that the exact **E** and **B** fields obtained above for the round wire in fact satisfy Maxwell's equations. Since the **B** equations were obtained from the curl **E** Maxwell equation, this one is not verified. The other three Maxwell equations are projected into their partial wave versions analogous to (D.4.6) above :

$$\text{div } \mathbf{B}(r,m) = r^{-1}\partial_r(rB_r) + r^{-1}\partial_\theta B_\theta + \partial_z B_z$$
$$= r^{-1}\partial_r(rB_r) + r^{-1}jmB_\theta - jk\,B_z \qquad \text{(D.5.1)}$$

$$\text{div } \mathbf{E}(r,m) = r^{-1}\partial_r(rE_r) + r^{-1}\partial_\theta E_\theta + \partial_z E_z$$
$$= r^{-1}\partial_r(rE_r) + r^{-1}jmE_\theta - jkE_z \qquad \text{(D.5.2)}$$

$$\text{curl } \mathbf{B}(r,m) = \hat{\mathbf{r}}[r^{-1}\partial_\theta B_z - \partial_z B_\theta] + \hat{\boldsymbol{\theta}}[\partial_z B_r - \partial_r B_z] + \hat{\mathbf{z}}[r^{-1}\partial_r(rB_\theta) - r^{-1}\partial_\theta B_r]$$
$$= \hat{\mathbf{r}}[r^{-1}jmB_z + jkB_\theta] + \hat{\boldsymbol{\theta}}[-jkB_r - \partial_r B_z] + \hat{\mathbf{z}}[r^{-1}\partial_r(rB_\theta) - r^{-1}jmB_r] \ . \qquad \text{(D.5.3)}$$

Inside the round wire we expect to find

div **B** = 0 // since **B** = (1/jω) curl **E** so div **B** = (1/jω) div curl **E** = 0
div **E** = 0 // no free charge

curl **B** = μ **J** + μ jωε**E** = μ(σ + jωε) **E** = μ(jω)(ε - jσ/ω) **E** = jω μξ **E**
= j (β^2/ω) **E** . // see (1.5.1c)

Thus, for the divergence equations we just compute the divergence as shown and see if it comes out zero, while for the curl **B** equation we verify that

curl **B** - j(β^2/ω) **E** = 0 (D.5.4)

for each component. The fact (D.2.2) that $\beta'^2 = \beta^2 - k^2$ is also used.

Here then is the Maple code which does the verification of the three Maxwell equations:

```
Show that div B  = 0 using (D.5.1)
divB := (1/r)*diff(r*Br,r)+(1/r)*j*m*Bphi-j*k*Bz: simplify(%);
                                                  0
Show that div E  = 0 using (D.5.2)
divE := (1/r)*diff(r*Er,r)+(1/r)*j*m*Ephi-j*k*Ez: simplify(%);
                                                  0
Compute components of curl B using (D.5.3)
curlB_r := (1/r)*j*m*Bz + j*k*Bphi:
curlB_phi := -j*k*Br - value(diff(Bz,r)):
curlB_z := value((1/r)*diff(r*Bphi,r)) - (1/r)*j*m*Br:
b := sqrt(bp^2+k^2):    set in beta here called b
One component at a time, show that the curl B Maxwell equation (D.5.4) is satsified
curlB_z - (j/w)*b^2*Ez: simplify(%);
                                                  0
curlB_r - (j/w)*b^2*Er: simplify(%);
                                                  0
curlB_phi - (j/w)*b^2*Ephi: simplify(%);
                                                  0
```

In Maple % refers to the last quantity computed. Prior to each simplify(%) statement we find a huge mess for the expression at hand, but simplify then shows it is really zero. As an example, here is the execution of the verification that [curl $\mathbf{B}]_z$ - j(β^2/ω) E_z = 0 :

```
> curlB_z - (j/w)*b^2*Ez; simplify(%);
```

$$\frac{1}{2}\left(2k^2\,am\left(-\mathrm{J}(m+1,bp\,r)+\frac{m\,\mathrm{J}(m,bp\,r)}{bp\,r}\right)bp+k^2\,Km\left(\mathrm{J}(m,bp\,r)-\frac{(m+1)\,\mathrm{J}(m+1,bp\,r)}{bp\,r}\right)bp^2\,r\right.$$
$$+k^2\,Km\,\mathrm{J}(m+1,bp\,r)\,bp+bp^4\,Km\left(\mathrm{J}(m,bp\,r)-\frac{(m+1)\,\mathrm{J}(m+1,bp\,r)}{bp\,r}\right)r+bp^3\,Km\,\mathrm{J}(m+1,bp\,r)$$
$$\left.-bp^3\,Km\,m\left(-\mathrm{J}(m+1,bp\,r)+\frac{m\,\mathrm{J}(m,bp\,r)}{bp\,r}\right)\right)/(r\,w\,k\,bp)$$
$$-\frac{1}{2}\frac{-m^2\,bp^2\,Km\,\mathrm{J}(m,bp\,r)+2k^2\,am\,\mathrm{J}(m,bp\,r)\,m-k^2\,\mathrm{J}(m+1,bp\,r)\,bp\,r\,Km\,m-2k^2\,\mathrm{J}(m+1,bp\,r)\,bp\,r\,am}{r^2\,w\,k\,bp}$$
$$-\frac{1}{2}\frac{(bp^2+k^2)\,bp\,Km\,\mathrm{J}(m,bp\,r)}{w\,k}$$
$$0$$

No approximations were made in the E fields, the B fields, or in these Maxwell verifications.

D.6 The exact E and B fields for the m=0 partial wave

The m=0 partial wave is all there is for an axially symmetric problem like that considered in Chapter 2, where the round wire is imagined in isolation, but is operationally the central conductor of a coaxial cable with a very distant return cylinder (outer shield). Here is the reduction of box (D.4.9) for m = 0, making use of the m=0 coefficients noted in box (D.2.28), namely,

$$m = 0 \qquad a_0 = 0 \qquad \left(\frac{K_0}{2}+\frac{a_0}{0}\right) = 0 \qquad\qquad \frac{K_0}{2} = (j\omega/\sigma)\,N_0\,\frac{1}{J_1(x_a)}$$

Summary of E and B fields inside a round wire (m = 0 only) (D.6.1)

$E_z(r,0) = -j\,(\beta'/k)\,\frac{K_0}{2}\,J_0(x)$ // large $\qquad x = \beta' r \qquad \beta'^2 = \beta^2 - k^2$

$E_r(r,0) = \frac{K_0}{2}\,J_1(x)$. // small $\qquad \frac{K_0}{2} = (j/2)\,(ak)\,I\,R_{dc}\,\frac{1}{J_1(x_a)}$

$jE_\theta(r,0) = 0 \qquad R_{dc} = \frac{1}{\sigma\pi a^2}$

$B_z(r,0) = 0$

$B_r(r,0) = 0$

$B_\theta(r,0) = (\beta'/\omega)\left(\frac{k}{\beta'}+\frac{\beta'}{k}\right)\frac{K_0}{2}\,J_1(x)$ // large $\qquad \sim \beta'\,(\beta'/k)$

No approximations have been made in these results, but a very good approximation for a low loss line is that $|\beta| >> |k| = |\beta_d|$ which means $\beta' \approx \beta$, as discussed below equation (D.2.2). With this approximation, we have commented in the above box on the size of the various field components. The dominant components are

$$E_z(r,0) = -j(\beta/k)\frac{K_0}{2} J_0(x) = -j(\beta/k)(j/2)(ak) I R_{dc} \frac{J_0(x)}{J_1(x_a)}$$

$$= (1/2)\beta a I R_{dc} \frac{J_0(x)}{J_1(x_a)} = (\omega/\beta)(1/2)(a\beta^2/\omega) I R_{dc} \frac{J_0(x)}{J_1(x_a)}$$

$$B_\theta(r,0) = (\beta/\omega)\left(\frac{\beta}{k}\right)(j/2)(ak) I R_{dc} \frac{J_1(x)}{J_1(x_a)}$$

$$= (j/2)(a\beta^2/\omega) I R_{dc} \frac{J_1(x)}{J_1(x_a)}.$$

As shown in (2.2.3) we can write $\beta^2/\omega \approx -j\mu\sigma$ so that

$$(1/2)(a\beta^2/\omega) I R_{dc} = (1/2) a (-j\mu\sigma) I \frac{1}{\sigma\pi a^2} = -j\frac{\mu I}{2\pi a}$$

and then the dominant components above become

$$E_z(r,0) = (\omega/\beta)\left[-j\frac{\mu I}{2\pi a}\right] \frac{J_0(x)}{J_1(x_a)} = -j(\omega/\beta)\frac{\mu I}{2\pi a}\frac{J_0(x)}{J_1(x_a)}$$

$$B_\theta(r,0) = j\left[-j\frac{\mu I}{2\pi a}\right]\frac{J_1(x)}{J_1(x_a)} = \frac{\mu I}{2\pi a}\frac{J_1(x)}{J_1(x_a)}. \qquad \text{(D.6.2)}$$

These results are in agreement with E(r) and B(r) shown in summary box (2.2.30) from the Chapter 2 calculation where we assumed $\mathbf{E} = E(r)\,\hat{\mathbf{z}}$ and $\mathbf{B} = B(r)\,\hat{\boldsymbol{\theta}}$