D.11 Low frequency limit of the round wire E fields

This has been completely replaced by a new Section D.11.

Our assumed Appendix D wave z dependence is e^{-jkz} from (D.1.1).

(a) An Inaccurate Model for Low Frequency

Recall from Chapter 4 that *for high frequencies* we derived the transmission line equations,

$$\frac{dV(z)}{dz} = -\mathbf{z}\, i(z) \qquad \frac{di(z)}{dz} = -\mathbf{y}\, V(z) \qquad (4.11.14b)$$

and from those we derived the second order uncoupled transmission line equations,

$$\frac{d^2V(z)}{dz^2} - \mathbf{zy}\, V(z) = 0 \qquad \frac{d^2i(z)}{dz^2} - \mathbf{zy}\, i(z) = 0 \quad . \qquad (4.11.15)$$

These last equations imply the following z-dependence for V(z) and i(z) and then presumably for all other transmission line non-power quantities (like the fields),

$$e^{-jkz} \qquad \text{where} \qquad k = -j\sqrt{\mathbf{zy}} = -j\sqrt{(R+j\omega L)(G+j\omega C)}$$

Summary of the E field solutions : $\qquad R_{dc} = \frac{1}{\sigma\pi a^2} \qquad \beta'^2 = \beta^2 - k'^2 \qquad (D.2.33)$

$$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (ak)\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

$$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (ak)\, h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$$

For large ω, we show in Appendix Q that

$$\text{For large } \omega: \quad \mathrm{Re}(\beta'_d) = \omega\sqrt{LC}$$

$$\mathrm{Im}(\beta'_d) = -\frac{RC+GL}{2\sqrt{LC}} \qquad \omega >> (R/L) \text{ and } \omega >> (G/C)$$

Since the wave z dependence is

$$e^{-jk'z} = e^{-j\mathrm{Re}(k')z}\; e^{-[-\mathrm{Im}(k')]z}$$

we can interpret Re(β'_d) as the "wavenumber" of the wave, and we know that this wavenumber = ω/v_d where v_d is the phase velocity of the wave. Thus, we identify

$$v_d = 1/\sqrt{LC}$$

in terms of which

$$\mathrm{Re}(\beta'_d) = \omega/v_d = k$$
$$-\mathrm{Im}(\beta'_d) = (RC+GL)(v_d/2) .$$

This last quantity is the attenuation parameter sometimes called α, so

$$\alpha = (RC+GL)(v_d/2) .$$

Thus we have a reasonable model for high frequency waves on our transmission line, including the attenuation due to the losses caused by R and G. We can go on to evaluate β'^2 and take its large ω limit,

$$\beta'^2 = \beta^2 - k'^2 = -j\mu\sigma\omega - [\omega/v_d - j\alpha]^2 \rightarrow -j\mu\sigma\omega - (\omega/v_d)^2 \approx -j\mu\sigma\omega .$$

In the last equality, although it seems the ω^2 term should win out at large ω, we have shown elsewhere that for any ω of transmission line interest, σ is large and v_d is large so $-(\omega/v_d)^2$ can be ignored. For example :

$f = 1000$ GHz $\qquad\qquad \omega = 2\pi \times 10^{12}$ Hz

$$\mu_0\sigma\omega = 4\pi \times 5.81 \times 2\pi \times 10^{12} = .46 \times 10^{15}$$

$$(\omega/v_d)^2 \approx (\omega/c)^2 = [2\pi \times 10^{12}/ 3\times10^8]^2 = [(2\pi/3) \times 10^4]^2 = .44 \times 10^9$$

In Section D.10 we went on to take the large β' limit of the f_m, g_m and h_m functions. This led to an interesting asymmetric J_z distribution which we associated with the proximity effect.

<u>For small ω</u> things are much more hazy. The main issue is that the transmission line equations quoted above are only justified from Maxwell's equations at high frequencies where the skin effect is at least starting to set in. We show in Appendix K that for the *network* model of a transmission line, the transmission line equations are valid at *all* ω. One must keep in mind that this network model is just a model, and we have shown in Chapter 4 that it does not correspond to the real world at low frequencies.

Despite this caveat, we can pretend that the transmission line equations *are* accurate at low ω and see what happens. We quote again from Appendix Q for a transmission line with G = 0 (vacuum dielectric) since this is assumed in our boundary condition (D.2.24),

For small ω: $\mathrm{Re}(\beta'_d) = \sqrt{\omega RC/2} = \alpha$ // a new α

$-\mathrm{Im}(\beta'_d) = \sqrt{\omega RC/2} = \alpha$ G = 0 and $\omega << (R/L)$

This is an extremely lossy wave, since it damps out in a half wave cycle. We then have

$$e^{-jk'z} = e^{-jRe(k')z}\ e^{-[-Im(k')]z} = e^{-j\alpha z}\, e^{-\sigma z} = e^{-(j+1)\alpha z} \quad .$$

This has the exact form shown in (2.1.8) so what we have here is in effect a longitudinal skin effect where the wave is trying to travel down the transmission line but it can't get very far. The longitudinal skin depth is $1/\alpha$.

We can blindly proceed and compute

$$\beta'^2 = \beta^2 - k'^2 = -j\mu\sigma\omega - \omega RC/2 \qquad \rightarrow\ 0 \ \text{ as } \omega\rightarrow 0$$

In this case, we want to take the small argument limits of f_m, g_m and h_m. We shall go ahead and do that below just for the sake of "completeness", but it should be realized that the results are not accurate.

(b) Low frequency evaluation of f_m, g_m and h_m and the E fields

In the following we consider only $m \geq 0$ since we know from (D.10.2) that $f_{-m} = f_m$, $g_{-m} = g_m$, $h_{-m} = h_m$.

The small x limit for $J_m(x)$ is given by NIST 10.7.3,

$$J_n(x) = (x/2)^n / n! \quad . \qquad \text{for } n = 0,1,2,..... \qquad \text{(D.11.14)}$$

Since J_{m-1} appears in our coefficient expressions and since m = 0 is encountered, we have to deal with m = 0 as a special case since the above limit is not valid for n = -1. To this end we use NIST 10.2.2 which is valid for integer n,

$$J_{-n}(x) = (-1)^n J_n(x) \approx (-1)^n (x/2)^n / n! \qquad \text{(D.11.15)}$$

so that $J_{-1}(x) = - J_1(x) \approx - (x/2)$. Our small-x forms of interest are then

$$\begin{aligned} &J_n(x) = (x/2)^n / n! && \text{for } n = 0,1,2,..... \\ &J_{-1}(x) = - (x/2) && \text{for } n = -1 \quad . \end{aligned} \qquad \text{(D.11.16)}$$

We now examine the small x limits of f_m, g_m, and h_m .

First f_m for m > 0, and then for m = 0:

$$f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] = \left[\frac{(x/2)^m/m!}{(x_a/2)^{m+1}/(m+1)!} - \frac{(x/2)^m/m!}{(x_a/2)^{m-1}/(m-1)!}\right]$$

$$= \left[(m+1)\,(x/x_a)^m\,(2/x_a) - (1/m)\,(x/x_a)^m (x_a/2) \right]$$

$$= (x/x_a)^m \left[(m+1)\,(2/x_a) - (1/m)\,(x_a/2) \right]$$

$$\approx \; (x/x_a)^m\,(m+1)\,(2/x_a) \qquad\qquad // \text{ as } x_a \to 0$$

$$f_0 = \left[\frac{J_0(x)}{J_1(x_a)} - \frac{J_0(x)}{J_{-1}(x_a)}\right] = \left[\frac{J_0(x)}{J_1(x_a)} + \frac{J_0(x)}{J_1(x_a)}\right] = 2\,\frac{J_0(x)}{J_1(x_a)} = 2\,\frac{1}{(x_a/2)} = 4/x_a$$

First g_m for m > 0, and then for m = 0:

$$g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] = \left[\frac{(x/2)^{m+1}/(m+1)!}{(x_a/2)^{m+1}/(m+1)!} + \frac{(x/2)^{m-1}/(m-1)!}{(x_a/2)^{m-1}/(m-1)!}\right]$$

$$= (x/x_a)^{m+1} + (x/x_a)^{m-1}$$

$$g_0 = \left[\frac{J_1(x)}{J_1(x_a)} + \frac{J_{-1}(x)}{J_{-1}(x_a)}\right] = \left[\frac{J_1(x)}{J_1(x_a)} + \frac{J_1(x)}{J_1(x_a)}\right] = 2\,\frac{J_1(x)}{J_1(x_a)} = 2\,(x/x_a)$$

Results for h_m are then obvious since there is only a sign change between the terms in g_m,

$$h_m = (x/x_a)^{m+1} - (x/x_a)^{m-1} \qquad\qquad h_0 = 0$$

The results are then (we replace k → k' as discussed in section (a) above) :

Small ω limit of the E field solutions : $\qquad R_{dc} = \dfrac{1}{\sigma\pi a^2}$ (D.11.17)

$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta')\, f_m$	$f_m = (r/a)^m\,(m+1)\,(2/\beta' a)$	$f_0 = 4/(a\beta')$
$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (ak')\, g_m$	$g_m = (r/a)^{m+1} + (r/a)^{m-1}$	$g_0 = 2\,(r/a)$
$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (ak')\, h_m$	$h_m = (r/a)^{m+1} - (r/a)^{m-1}$	$h_0 = 0$

For m > 0:

$E_z(r,m) = (1/2)\,\eta_m\, I\, R_{dc}\, (r/a)^m\,(m+1)$

$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (ak')\, [(r/a)^{m+1} + (r/a)^{m-1}]$

$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (ak')\, [(r/a)^{m+1} - (r/a)^{m-1}]$

For m= 0:

$E_z(r,0) = I\, R_{dc}$

$E_r(r,0) = (j/2)\, I\, R_{dc}\, (ak')\, (r/a)$

$E_\theta(r,0) = 0$

Recall now these field expansions for the case that n(θ) is an even function of θ,

$$E_z(r,\theta) = (1/4)\, I\, R_{dc}\, (a\beta')\, [\, f_0 + 2\,\Sigma_{m=1}^{\infty}\, f_m\, \eta_m\, \cos(m\theta)\,]$$

$$E_r(r,\theta) = (j/4)\, I\, R_{dc}\, (ak')\, [\, g_0 + 2\, \Sigma_{m=1}^{\infty}\, g_m\, \eta_m\, \cos(m\theta)\,]$$

$$E_\theta(r,\theta) = (1/4)\, I\, R_{dc}\, (ak')\, [\, h_0 + 2\, \Sigma_{m=1}^{\infty}\, h_m\, \eta_m\, \cos(m\theta)\,]\ . \qquad \textit{(D.10.4a)}$$

For low ω we insert the expressions above to get

$$E_z(r,\theta) = I\, R_{dc}\, [\, 1 + \Sigma_{m=1}^{\infty}\, (r/a)^m\, (m+1)\, \eta_m\, \cos(m\theta)\,]$$

$$E_r(r,\theta) = (j/2)\, I\, R_{dc}\, (ak)\, [\, (r/a) + \Sigma_{m=1}^{\infty}\, [(r/a)^{m+1} + (r/a)^{m-1}]\, \eta_m\, \cos(m\theta)\,]$$

$$E_\theta(r,\theta) = (1/2)\, I\, R_{dc}\, (ak)\, [\, \Sigma_{m=1}^{\infty}\, [(r/a)^{m+1} - (r/a)^{m-1}]\, \eta_m\, \cos(m\theta)\,]\ . \qquad \text{(D.11.18)}$$

THIS SECTION ABOVE IS ON HOLD

Observations on the E fields for small ω

2. The m = 0 term in E_z is just $E_z = IR_{dc}$ which says $J_z = \sigma\, IR_{dc} = \sigma\, I(1/\sigma\pi a^2) = I/(\pi a^2)$. This is the current one would expect in a wire carrying DC current I.

3. Assuming a non-uniform n(θ) charge density on the wire surface, the η_m moments are non-zero and one concludes that $J_z(r,\theta)$ is asymmetric even as $\omega \to 0$. This conclusion is incorrect, and the reason is that we have improperly assumed STOP

$$I = V/Z_0$$

and according to (4.11.16), where G = 0 since we assumed a non-conducting dielectric,

$$Z_0 \equiv V(z)/i(z) = \sqrt{\frac{\mathbf{z}}{\mathbf{y}}} = \sqrt{\frac{R + j\omega L}{j\omega C}} \ . \qquad (4.11.16)$$

Thus in the limit $\omega \to 0$ we get $Z_0 \to \infty$. For a finite length transmission line, as one lowers ω, one must increase the size of the termination Z_0 to maintain a properly terminated line. Thus, our situation here does not apply to taking the low frequency limit of a transmission line terminated by a fixed 75Ω or 8Ω resistor. See Section 6.5 (e) for more on this subject. The major point here is that the entire model is not valid for low ω and we cannot use it to study the limit $\omega \to 0$.