Rewrite Section 1.6 PhL 12.3.13

This painful doc was fully reviewed on 12.5.13, see comments in red. All issues are resolved.

(1) When I first wrote this section, I was aware of the fact that in the round wire example, the E and B fields would come out being complex in a complicated manner, and here I was trying to "allow for that" in my formalism. But now I realize that the round wire E and B fields are the **E**(r,m) type partial waves and not the actual E and B fields like **E**(**x**,t) . So that motivation is weakened. Especially section (b) was motivated in this manner, and I now think that section should be completely dropped. NO, belay that order, because I refer to this later in the round wire stuff with true **E**(**x**,t) fields!

The issue of different field components having different phases.

(2) In the original, I started with this breakdown where I allowed **E** to be complex

$$\mathbf{E}(x,t) = \mathbf{e}(x,t) + j\,\mathbf{e}'(x,t)$$

where now **e** and **e'** are real and hopefully are physical for different problems. I then made a mistake. I forced all components of **E**(**x**,t) to have the same phase in a monochrome situation, such as

$$\mathbf{E}(\mathbf{x},t) = e^{j[\omega_1 t+\varphi(\mathbf{x},\omega_1)]}\,\boldsymbol{e}(\mathbf{x},\omega_1)$$

where I clamed then that $\boldsymbol{e}(\mathbf{x},\omega_1)$ was real. That would mean in turn that

$$\mathbf{e}(\mathbf{x},t) = \mathrm{Re}\{\,\mathbf{E}(\mathbf{x},t)\} = \cos[\omega_1 t + \varphi(\mathbf{x},\omega_1)]\,\boldsymbol{e}(\mathbf{x},\omega_1)$$

I now think if you really want to write out all this detail, you have to say

$$E_i(\mathbf{x},t) = e^{j[\omega_1 t+\varphi_{Ei}(\mathbf{x},\omega_1)]}\,e_i(\mathbf{x},\omega_1) = e^{j\omega_1 t}\,e^{j\varphi_{Ei}(\mathbf{x},\omega_1)}\,e_i(\mathbf{x},\omega_1)\ , \qquad (1.6.5)$$

$$\mathbf{E}(x,t) = \mathbf{e}(x,t) + j\,\mathbf{e}'(x,t)$$

$$e_i(\mathbf{x},t) = \mathrm{Re}\{\,E_i(\mathbf{x},t)\} = \cos[\omega_1 t + \varphi_{Ei}(\mathbf{x},\omega_1)]\,e_i(\mathbf{x},\omega_1)$$

$$e'_i(\mathbf{x},t) = \mathrm{Im}\{\,E_i(\mathbf{x},t)\} = \sin[\omega_1 t + \varphi_{Ei}(\mathbf{x},\omega_1)]\,e_i(\mathbf{x},\omega_1) \quad . \qquad (1.6.6)$$

This then becomes extremely ugly, there is not even a reasonable vector notation! So I now wonder what if any is the benefit of "breaking out the phase" as I have done [no benefit!]

Well, in the end I decided is really WAS better to break out the phases in Section 1.6 and that is now all installed into lines doc.

Two Ansatz's that are nice but don't apply to the round wire

How could these be redone?

$$\mathrm{curl}\ \mathbf{E}(\mathbf{x},t) = -\,\partial\mathbf{B}(x,t)/\partial t \qquad \mathbf{E}(x,t) = \mathbf{e}(x,t) + j\,\mathbf{e}'(x,t)$$

$$\mathbf{B}(x,t) = \mathbf{b}(x,t) + j\,\mathbf{b}'(x,t) \quad . \qquad \text{ok} \tag{1.6.3}$$

Is this OK or not? I imagine that E and B are complex, and that all lower case are real.

$$\begin{aligned} &\text{curl } \mathbf{e}(\mathbf{x},t) = -\,\partial\mathbf{b}(x,t)/\partial t \\ &\text{curl } \mathbf{e}'(\mathbf{x},t) = -\,\partial\mathbf{b}'(x,t)/\partial t \quad . \qquad \text{ok} \end{aligned} \tag{1.6.4}$$

This last seems "still true". Now part (b) would just say [this is a possible ansatz, call it #1]

$$\mathbf{E}(\mathbf{x},t) = e^{j\omega_1 t}\,\boldsymbol{E}(\mathbf{x},\omega_1) \quad \Rightarrow \quad \begin{aligned} &\mathbf{e}(x,t) = \cos(\omega_1 t)\,\boldsymbol{E}(\mathbf{x},\omega_1) \\ &\mathbf{e}'(x,t) = \sin(\omega_1 t)\,\boldsymbol{E}(\mathbf{x},\omega_1) \end{aligned}$$

It seems now that $\boldsymbol{E}(\mathbf{x},\omega_1)$ must be real [that statement is part of the ansatz] because at the start **e** and **e'** were real by definition. So I don't have to extract some phase from $\boldsymbol{E}(\mathbf{x},\omega_1)$ in order to get $\boldsymbol{E}(\mathbf{x},\omega_1)$ to be real. I was being distracted by the round wire stuff when I did this.

But, consider this result from Chapter 2:

$$\mathbf{E}(\mathbf{x},t) = E_0\, e^{j\omega t}\, \frac{\mathrm{ber}_0[r\sqrt{\omega\mu\sigma}\,] + j\,\mathrm{bei}_0[r\sqrt{\omega\mu\sigma}\,]}{\mathrm{ber}_0[a\sqrt{\omega\mu\sigma}\,] + j\,\mathrm{bei}_0[a\sqrt{\omega\mu\sigma}\,]}\,\hat{\mathbf{z}} \tag{2.3.14}$$

This does not seem to fit into the above plan because here $\boldsymbol{E}(\mathbf{x},\omega_1)$ is complex!! Here the phase of what would be called $\boldsymbol{E}(\mathbf{x},\omega_1)$ varies with position **x** and/or with frequency ω. So the round wire does not admit the above ansatz. Notice that complex $\boldsymbol{E}$ is also compatible with (1.6.3),

$$\mathbf{E}(\mathbf{x},t) = e^{j\omega_1 t}\,\boldsymbol{E}(\mathbf{x},\omega_1) = e^{j\omega_1 t}\,\{\,\mathrm{Re}(\boldsymbol{E}) + j\mathrm{Im}(\boldsymbol{E})\,\} = \text{real part} + j\,\text{imaginary part for 1.6.3}$$

Now in Appendix D I am going to say

$$\mathbf{E}(\mathbf{x},t) = e^{j(\omega t - kz)}\,\boldsymbol{E}(\mathbf{x},\omega_1) \quad \begin{aligned} &\Rightarrow \quad \mathbf{e}(x,t) = \cos(\omega t - kz)\,\boldsymbol{E}(\mathbf{x},\omega) \\ &\Rightarrow \quad \mathbf{e}'(x,t) = \sin(\omega t - kz)\,\boldsymbol{E}(\mathbf{x},\omega) \end{aligned}$$

Although this is now a different $\boldsymbol{E}(\mathbf{x},\omega_1)$, it is still REAL! [no, it is just an ansatz that won't work in App D, just as above for the round wire.] The implication is that all components of the E field have the SAME phase which is (ωt-kz). [but the real world solution has an extra position-dependent term in this phase and each component has a different phase] Is that what I want for wave action? Yes it is. I don't want the different components to have different k_i wavenumbers. At any rate, by ansatz I see solutions of this form. [Yes, it is just an ansatz, and just because you want something to work a certain way does not make it work that way.]

below is already out of date!

This is basically my old original Section 1.6 where I played with modifying the first part (b). Today 12/5/13 I rewrite Section 1.6 to include the separate phases and mode some other repairs. Therefore, you can completely ignore what is below.

1.6 Reinterpretation of all equations in terms of complex functions

It seemed useful to defer the topics of this section to avoid cluttering up the preceding five sections. The Fourier Transform has already been used in the previous two sections, and here we shall discuss it more formally as a motivating factor in changing our point of view from real to complex functions. The general nature of the Fourier transform of complex monochrome ($e^{j\omega t}$) fields sets the stage for the analysis of the round wire in Section 2.

(a) Complex Functions

Up to this point, we have been regarding the following fields as representing real physical quantities,

$$\begin{array}{llll} \mathbf{H}(\mathbf{x},t) & \mathbf{D}(\mathbf{x},t) & \mathbf{J}(\mathbf{x},t) & \mathbf{A}(\mathbf{x},t) \\ \mathbf{B}(\mathbf{x},t) & \mathbf{E}(\mathbf{x},t) & \rho(\mathbf{x},t) & \varphi(\mathbf{x},t) \end{array} \qquad (1.6.1)$$

The fields, potentials and sources exist in the real physical world and are related by equations involving real operators like curl and $\partial/\partial t$. We can represent such an equation as $L_{\mathbf{x},t} f(\mathbf{x},t) = g(\mathbf{x},t)$ where $L_{\mathbf{x},t}$ is some real differential operator and f and g are real fields (g does not mean Green's function here).

One can extend f and g such that f and g are either both the real or both the imaginary parts of complex functions F and G. Then the equation $L_{\mathbf{x},t} F(\mathbf{x},t) = G(\mathbf{x},t)$ represents two distinct physical equations which we can write as

$$L_{\mathbf{x},t} F(\mathbf{x},t) = G(\mathbf{x},t) \quad \Rightarrow \quad L_{\mathbf{x},t}[f(\mathbf{x},t) + jf'(\mathbf{x},t)] = [g(\mathbf{x},t) + jg'(\mathbf{x},t)] \quad \Rightarrow$$

$$\begin{array}{ll} L_{\mathbf{x},t}\, f(x,t) = g(x,t) & F(\mathbf{x},t) = f(\mathbf{x},t) + jf'(\mathbf{x},t) \\ L_{\mathbf{x},t}\, f'(x,t) = g'(x,t) & G(\mathbf{x},t) = g(\mathbf{x},t) + jg'(\mathbf{x},t)\ . \end{array} \qquad (1.6.2)$$

It is convenient to regard all the mathematical fields listed above in (1.6.1) as complex fields like F and G. For example, we might write the Maxwell curl **E** equation (1.1.2) in this manner

$$\begin{array}{ll} \text{curl } \mathbf{E}(\mathbf{x,t}) = -\,\partial \mathbf{B}(x,t)/\partial t & \mathbf{E}(x,t) = \mathbf{e}(x,t) + j\,\mathbf{e}'(x,t) \\ & \mathbf{B}(x,t) = \mathbf{b}(x,t) + j\,\mathbf{b}'(x,t)\ . \end{array} \qquad (1.6.3)$$

where $\mathbf{e} = \mathrm{Re}(\mathbf{E})$ and $\mathbf{e}' = \mathrm{Im}(\mathbf{E})$ and similarly for the **B** field.

The single left equation of (1.6.3) then represents these two different physical equations with real fields

$$\begin{array}{l} \text{curl } \mathbf{e(x,t)} = -\,\partial \mathbf{b}(x,t)/\partial t \\ \text{curl } \mathbf{e'(x,t)} = -\,\partial \mathbf{b'}(x,t)/\partial t\ . \end{array} \qquad (1.6.4)$$

ok to here

(b) Monochrome time

The classic application of this idea is the assumption that some complex field is "monochrome" in its time dependence, meaning for example,

$$\mathbf{E}(\mathbf{x},t) = e^{j[\omega_1 t+\varphi(\mathbf{x},\omega_1)]}\, \boldsymbol{E}(\mathbf{x},\omega_1) \;=\; e^{j\omega_1 t}\, e^{j\varphi(\mathbf{x},\omega_1)}\, \boldsymbol{E}(\mathbf{x},\omega_1) \;, \qquad (1.6.5)$$

where $\boldsymbol{E}(\mathbf{x},\omega_1)$ is real. All time dependence is in the $e^{j\omega_1 t}$ factor and all spatial dependence is in the factor $[e^{j\varphi(\mathbf{x},\omega_1)}\, \boldsymbol{E}(\mathbf{x},\omega_1)]$ -- separation of variables. This monochrome field might be regarded as a probe or driver of some system and the solution function $\boldsymbol{E}(\mathbf{x},\omega_1)$ and phase $\varphi(\mathbf{x},\omega_1)$ might depend parametrically on the probe frequency ω_1 as well as on position $\mathbf{x}$.

OK, here in (1.6.5) I make the ansatz that all components of E have the same phase function. That is a fine ansatz, but does not work for the round wire.

For (1.6.5) the corresponding physical field assumption is either of these equations,

$$\begin{aligned} \mathbf{e}(\mathbf{x},t) &= \mathrm{Re}\{\, \mathbf{E}(\mathbf{x},t)\} = \cos[\omega_1 t + \varphi(\mathbf{x},\omega_1)]\, \boldsymbol{E}(\mathbf{x},\omega_1) \\ \mathbf{e}'(\mathbf{x},t) &= \mathrm{Im}\{\, \mathbf{E}(\mathbf{x},t)\} = \sin[\omega_1 t + \varphi(\mathbf{x},\omega_1)]\, \boldsymbol{E}(\mathbf{x},\omega_1) \quad . \end{aligned} \qquad (1.6.6)$$

A good prototype example for the ω_1 dependence of phase $\varphi(\mathbf{x},\omega_1)$ is a damped harmonic oscillator with resonant frequency ω_0 which is driven at frequency ω_1. The solution is:

$$x(t) = x(0)\, \sin[\omega_1 t + \varphi(\omega_1)] \qquad \tan\varphi(\omega_1) = -(\omega_1/\tau)/(\omega_0^2 - \omega_1^2)$$

Of course the solution function x(t) is not a field over R^3, so in this case the phase φ has no $\mathbf{x}$ dependence.

Comments: [all correct, but these ansatz types don't work for the round wire.]

The assumed form (1.6.5) is *not* the most general form possible for monochromatic $\mathbf{E}(\mathbf{x},t)$. [correct] One could, for example, allow each field component $E_i(\mathbf{x},t)$ to have a separate phase $\varphi_i(\mathbf{x},t)$. The form (1.6.5) is really an "ansatz" form meaning that one assumes an $\mathbf{E}$ field of form (1.6.5), and then one tries to find the solution to a specific problem with the $\mathbf{E}$ field so restricted. If a solution is found, then the assumed form is justified. Often one makes a more restrictive ansatz where $\varphi(\mathbf{x},\omega) = \varphi(\omega)$, or even more restrictive with $\varphi(\mathbf{x},\omega) = \varphi$, a constant, perhaps $\varphi = 0$.

For a different kind of problem, one might make an ansatz different from (1.6.5). One might for example write $\mathbf{E}(\mathbf{x},t) = e^{j(\omega_1 t - kx)}\, \boldsymbol{E}(\mathbf{x},\omega_1)$ if one were searching for a solution representing some kind of wave traveling in the x direction. This $\boldsymbol{E}(\mathbf{x},\omega_1)$ is then different from that shown in (1.6.5).

(c) Why complex fields?

The reason for using a complex field like $\mathbf{E}(\mathbf{x},t)$ instead of the real field $\mathbf{e}(\mathbf{x},t)$ has to do with the Fourier Transform (or the Laplace Transform). This transform is almost always needed to solve a non-trivial problem involving Maxwell's equations, and we saw it in action in Section 1.5. In our somewhat sloppy notation, and with the convention that the $(1/2\pi)$ goes in the expansion formula along with $e^{+j\omega t}$, we write this transform as :

$$\mathbf{E}(\mathbf{x},\omega) = \int_{-\infty}^{\infty} dt\ \mathbf{E}(\mathbf{x},t)\ e^{-j\omega t} \qquad \text{projection = transform} \qquad (1.6.7)$$

$$\mathbf{E}(\mathbf{x},t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\ \mathbf{E}(\mathbf{x},\omega)\ e^{+j\omega t}\ . \qquad \text{expansion = inverse transform = recovery} \qquad (1.6.8)$$

Here $\mathbf{E}(\mathbf{x},t)$ is the original complex field whose real and imaginary parts are physical fields as in (1.6.3) or (1.6.6), while $\mathbf{E}(\mathbf{x},\omega)$ is the Fourier Transform of $\mathbf{E}(\mathbf{x},t)$. Since $\mathbf{E}(\mathbf{x},\omega)$ is a completely different complex function from $\mathbf{E}(\mathbf{x},t)$, one really should use some notation like $\boldsymbol{\mathcal{E}}(\mathbf{x},\omega)$ or $\mathbf{E}\hat{}(\mathbf{x},\omega)$, but we trust the reader to make the distinction when the ω argument is present or in the general context of some discussion.

As (1.6.7) shows, the dimensional units of the Fourier transform of some quantity have an extra sec factor. For example, since dim[$\mathbf{E}(\mathbf{x},t)$] = volt/m, it follows that dim[$\mathbf{E}(\mathbf{x},\omega)$] = volt-sec/m.

An obvious property of the Fourier Transform is this:

$$\partial_t \mathbf{E}(\mathbf{x},t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\ \mathbf{E}(\mathbf{x},\omega)\ \partial_t\ e^{+j\omega t} = (1/2\pi) \int_{-\infty}^{\infty} d\omega\ [j\omega\ \mathbf{E}(\mathbf{x},\omega)]\ e^{+j\omega t}$$

which we can write as (symbol ↔ means "corresponds to")

$$\mathbf{E}(\mathbf{x},t) \leftrightarrow \mathbf{E}(\mathbf{x},\omega) \qquad \Leftrightarrow \qquad \partial_t \mathbf{E}(\mathbf{x},t) \leftrightarrow\ j\omega\ \mathbf{E}(\mathbf{x},\omega) \qquad (1.6.9)$$

which is just another way to state our rule (1.5.2). In the case of assumed monochrome time dependence of the form (1.6.5) (reflected in (1.6.6)) one finds that

$$\mathbf{E}(\mathbf{x},t) = e^{j[\omega_1 t+\varphi(\mathbf{x},\omega_1)]}\ \boldsymbol{E}(\mathbf{x},\omega_1) \qquad \textit{(1.6.5)}$$

$$\mathbf{E}(\mathbf{x},\omega) = \int_{-\infty}^{\infty} dt\ [e^{j\omega_1 t}\ e^{j\varphi(\mathbf{x},\omega_1)}\boldsymbol{E}(\mathbf{x},\omega_1)]\ e^{-j\omega t} = \boldsymbol{E}(\mathbf{x},\omega_1)\ e^{j\varphi(\mathbf{x},\omega_1)} \int_{-\infty}^{\infty} dt\ e^{j(\omega_1-\omega)t}$$

$$= \boldsymbol{E}(\mathbf{x},\omega_1)\ e^{j\varphi(\mathbf{x},\omega_1)}\ 2\pi\delta(\omega-\omega_1)\ . \qquad (1.6.10)$$

It is this simple single-δ-function form that motivates the use of complex fields as carriers of the real physical fields. One can of course Fourier-transform the monochrome physical field directly, but the result is clumsy to deal with. For example,

$$\mathbf{e}(\mathbf{x},t) = \cos[\omega_1 t + \varphi_1(\mathbf{x},\omega_1)]\ \boldsymbol{E}(\mathbf{x},\omega_1)$$

$$\mathbf{e}(\mathbf{x},\omega) = \int_{-\infty}^{\infty} dt\ \{\cos[\omega_1 t + \varphi_1(\mathbf{x},\omega_1)]\ \boldsymbol{E}(\mathbf{x},\omega_1)\}e^{-j\omega t}$$

$$= \boldsymbol{E}(\mathbf{x},\omega_1)\ (1/2) \int_{-\infty}^{\infty} dt\ \{\ e^{j[\omega_1 t+\varphi(\mathbf{x},\omega_1)]} + e^{-j[\omega_1 t+\varphi(\mathbf{x},\omega_1)]}\ \}\ e^{-j\omega t}$$

$$= \boldsymbol{E}(\mathbf{x},\omega_1)\ [e^{j\varphi(\mathbf{x},\omega_1)}\pi\delta(\omega-\omega_1) + e^{-j\varphi(\mathbf{x},\omega_1)}\pi\delta(\omega+\omega_1)\ ]\ . \qquad (1.6.11)$$

A directly related benefit of using the complex function approach is the fact that math with exponentials is so much simpler than the corresponding math with trig functions, as for example

$$e^{j(\omega t+\varphi)}\, e^{-j(\omega' t+\varphi')} = e^{j(\omega-\omega')t}\, e^{j(\varphi-\varphi')} \qquad \text{// dependence on t isolated to one factor}$$

$$\cos(\omega t+\varphi)\cos(\omega' t+\varphi') = (1/2)\ \{\ \cos[\ (\omega-\omega')t + (\varphi-\varphi')]\ +\ \cos[\ (\omega+\omega')t + (\varphi+\varphi')]\ \}\ . \qquad (1.6.12)$$

Another benefit of using the Fourier transform is its close connection with the Laplace Transform.

Comment: Using the real cosine form shown as the first line of (1.6.6) along with the Fourier Cosine Transform is not viable because $\cos[\omega_1 t + \varphi(\mathbf{x},\omega_1)]\ \boldsymbol{E}(\mathbf{x},\omega_1)$ is not an even function of t.

(d) Monochrome E and B fields

One might seek to solve a system using monochrome fields of the form (1.6.5) for both the electric and magnetic fields. Those forms would be ($\boldsymbol{e}$ and $\boldsymbol{b}$ are real)

$$\mathbf{E}(\mathbf{x},t) = e^{j[\omega_1 t+\varphi_e(\mathbf{x},\omega_1)]}\ \boldsymbol{e}(\mathbf{x},\omega_1)$$
$$\mathbf{B}(\mathbf{x},t) = e^{j[\omega_1 t+\varphi_b(\mathbf{x},\omega_1)]}\ \boldsymbol{b}(\mathbf{x},\omega_1) \qquad (1.6.13)$$

where we assume the same frequency ω_1 for both fields, but allow the fields to have different phase functions φ_e and φ_b. In this case (1.6.10) becomes

$$\mathbf{E}(\mathbf{x},\omega) = \boldsymbol{e}(\mathbf{x},\omega_1)\ e^{j\varphi_e(\mathbf{x},\omega_1)}\ 2\pi\delta(\omega-\omega_1)$$
$$\mathbf{B}(\mathbf{x},\omega) = \boldsymbol{b}(\mathbf{x},\omega_1)\ e^{j\varphi_b(\mathbf{x},\omega_1)}\ 2\pi\delta(\omega-\omega_1)\ . \qquad (1.6.14)$$

Suppose the directions of the **E** and **B** fields are $\hat{\mathbf{n}}_e$ and $\hat{\mathbf{n}}_b$. Then (1.6.14) says

$$E_{n_e}(\mathbf{x},\omega) = e_{n_e}(\mathbf{x},\omega_1)\ e^{j\varphi_e(\mathbf{x},\omega_1)}\ 2\pi\delta(\omega-\omega_1)$$
$$B_{n_b}(\mathbf{x},\omega) = b_{n_b}(\mathbf{x},\omega_1)\ e^{j\varphi_b(\mathbf{x},\omega_1)}\ 2\pi\delta(\omega-\omega_1) \qquad (1.6.15)$$

and one finds that

$$\frac{E_{n_e}(\mathbf{x},\omega)}{B_{n_b}(\mathbf{x},\omega)} = \frac{e_{n_e}(\mathbf{x},\omega_1)}{b_{n_b}(\mathbf{x},\omega_1)}\ e^{j[\varphi_e(\mathbf{x},\omega_1)-\varphi_b(\mathbf{x},\omega_1)]}\ . \qquad (1.6.16)$$

Since $\boldsymbol{e}$ and $\boldsymbol{b}$ are real, the phase of the ratio E_{n_e}/B_{n_b} is determined by the last factor and will in general be a function of both position **x** and frequency ω_1. Again, this type of result will appear in the round wire analysis of the next section.

(e) A Pitfall to Avoid

Notice that

$$\mathbf{E}(\mathbf{x},t) = \mathbf{e}(\mathbf{x},t) + j\,\mathbf{e}'(\mathbf{x},t) \quad \Rightarrow$$

$$\mathbf{E}(\mathbf{x},\omega) = \int_{-\infty}^{\infty} dt\, \mathbf{E}(\mathbf{x},t)\, e^{-j\omega t} = \int_{-\infty}^{\infty} dt\, [\mathbf{e}(\mathbf{x},t) + j\,\mathbf{e}'(\mathbf{x},t)]\, e^{-j\omega t}$$

$$= \mathbf{e}(\mathbf{x},\omega) + j\,\mathbf{e}'(\mathbf{x},\omega)\,. \tag{1.6.17}$$

Whereas **e(x,t)** and **e'(x,t)** are the real and imaginary parts of **E**(**x**,t), the functions **e**(**x**,ω) and **e'**(**x**,ω) are *not* the real and imaginary parts of **E**(**x**,ω) since in general **e**(**x**,ω) and **e'**(**x**,ω) are both complex functions. In this document we shall never deal with transforms of the type **e**(**x**,ω) or **e'**(**x**,ω).

(f) Maxwell's Equations in ω space

In the Maxwell and related equations which include the ∂_t operator, if the fields are expanded onto their Fourier transformed components using (1.6.8), then using the rule (1.6.9) one may instantly write the frequency-domain version of these equations, just as in the example of Section 1.5. For example,

$$\text{curl } \mathbf{H}(\mathbf{x},\omega) = j\omega\mathbf{D}(\mathbf{x},\omega) + \mathbf{J}(\mathbf{x},\omega) \tag{1.6.18}$$

$$\text{curl } \mathbf{E}(\mathbf{x},\omega) = -j\omega\mathbf{B}(\mathbf{x},\omega) \tag{1.6.19}$$

$$\text{div } \mathbf{J}(\mathbf{x},\omega) = -j\omega\,\rho(\mathbf{x},\omega)\,. \tag{1.6.20}$$

Other equations in the Section 1.1 list have the same form but in terms of the frequency-domain functions. For example,

$$\mathbf{J}(\mathbf{x},\omega) = \sigma(\mathbf{x})\,\mathbf{E}(\mathbf{x},\omega) \tag{1.6.21}$$

where we momentarily allow σ(**x**) to have spatial dependence but not time dependence.

********** below are ponderings moved from the edit log **********

Question: In (D.1.2) can I really take **E**(r,φ) to be real? That would mean it always has a zero time phase shift in the sense of Chapter 1.6, all three components! In Chap 1.6 I said this

$$\mathbf{E}(\mathbf{x},t) = e^{j[\omega_1 t + \varphi(\mathbf{x},\omega_1)]}\,\boldsymbol{e}(\mathbf{x},\omega_1) = e^{j\omega_1 t}\, e^{j\varphi(\mathbf{x},\omega_1)}\,\boldsymbol{e}(\mathbf{x},\omega_1)\,, \tag{1.6.5}$$

where $\boldsymbol{e}(\mathbf{x},\omega_1)$ was real. Maybe even this is not general enough, because it forces all three components of the **E**(**x**,t) field to have the same phase! But then I say

$$\mathbf{e}(\mathbf{x},t) = \text{Re}\{\,\mathbf{E}(\mathbf{x},t)\} = \cos[\omega_1 t + \varphi(\mathbf{x},\omega_1)]\,\boldsymbol{e}(\mathbf{x},\omega_1)$$
$$\mathbf{e}'(\mathbf{x},t) = \text{Im}\{\,\mathbf{E}(\mathbf{x},t)\} = \sin[\omega_1 t + \varphi(\mathbf{x},\omega_1)]\,\boldsymbol{e}(\mathbf{x},\omega_1)\,. \tag{1.6.6}$$

and again all three components have the same temporal phase. This now seems wrong to me because one direction might be "resistive" and another direction "capacitative" in some problem, and then you would expect different phases!

Consider this curl E equation in the case that I allow 3 components to have different phases, and here I will do Cartesian

$$\text{curl } \mathbf{E}(\mathbf{x},t) = -\partial_t \mathbf{B}(\mathbf{x},t)$$

$$[\text{curl } \mathbf{E}(\mathbf{x},t)]_i = -\partial_t B_i(\mathbf{x},t)$$

$$E_k(\mathbf{x},t) = [e^{j[\omega_1 t+\varphi_{ek}(\mathbf{x},\omega_1)]}\, e_k(\mathbf{x},\omega_1)]$$
$$B_k(\mathbf{x},t) = e^{j[\omega_1 t+\varphi_{bk}(\mathbf{x},\omega_1)]}\, b_k(\mathbf{x},\omega_1) \qquad (1.6.13)$$

$$\varepsilon_{ijk}\, \partial_j E_k(\mathbf{x},t) = -\partial_t B_i(\mathbf{x},t)$$

$$\varepsilon_{ijk}\, \partial_j [e^{j[\omega_1 t+\varphi_{ek}(\mathbf{x},\omega_1)]}\, e_k(\mathbf{x},\omega_1)] = -\partial_t [e^{j[\omega_1 t+\varphi_{bi}(\mathbf{x},\omega_1)]}\, b_i(\mathbf{x},\omega_1)]$$

On the left side there are two terms!

$$\partial_j [e^{j[\omega_1 t+\varphi_{ek}(\mathbf{x},\omega_1)]}\, e_k(\mathbf{x},\omega_1)] = e^{j[\omega_1 t+\varphi_{ek}(\mathbf{x},\omega_1)]}\, e_k(\mathbf{x},\omega_1)\, j\, \partial_j \varphi_{ek} + e^{j[\omega_1 t+\varphi_{ek}(\mathbf{x},\omega_1)]} \partial_j\, e_k(\mathbf{x},\omega_1)$$

We then end up with

$$\varepsilon_{ijk}\, e^{j[\omega_1 t+\varphi_{ek}(\mathbf{x},\omega_1)]}\, e_k(\mathbf{x},\omega_1)\, j\, \partial_j \varphi_{ek} + e^{j[\omega_1 t+\varphi_{ek}(\mathbf{x},\omega_1)]} \partial_j\, e_k(\mathbf{x},\omega_1) = -j\omega\, [e^{j[\omega_1 t+\varphi_{bi}(\mathbf{x},\omega_1)]}\, b_i(\mathbf{x},\omega_1)]$$

or

$$\varepsilon_{ijk}\, e^{j[\varphi_{ek}(\mathbf{x},\omega_1)]}\, e_k(\mathbf{x},\omega_1)\, j\, \partial_j \varphi_{ek} + e^{j[\varphi_{ek}(\mathbf{x},\omega_1)]} \partial_j\, e_k(\mathbf{x},\omega_1) = -j\omega\, [e^{j[\varphi_{bi}(\mathbf{x},\omega_1)]}\, b_i(\mathbf{x},\omega_1)]$$

I see nothing here that forces all three φ_{ek} to be the same!

Similarly, suppose we have a wave equation in some region of space with no ρ or **J,** and again we are in Cartesian,

$$(\nabla^2 - \mu\varepsilon\, \partial_t^2) E_k = 0 \qquad (1.2.1)$$

$$(\nabla^2 - \mu\varepsilon\, \partial_t^2)\, [e^{j[\omega_1 t+\varphi_{ek}(\mathbf{x},\omega_1)]}\, e_k(\mathbf{x},\omega_1)] = 0$$

$$(\nabla^2 + \omega_1^2 \mu\varepsilon\,)\, [e^{j[\omega_1 t+\varphi_{ek}(\mathbf{x},\omega_1)]}\, e_k(\mathbf{x},\omega_1)] = 0$$

$$(\nabla^2 + \omega_1^2\mu\varepsilon)\,[e^{j[\varphi_{ek}(\mathbf{x},\omega_1)]}\, e_k(\mathbf{x},\omega_1)] = 0$$

Again ∇^2 acts on the phase as well as on e_k. I see nothing here forcing the thee φ_{ek} to be the same.

So I think this is a mistake in chapter 1.6.

Fact: You cannot write $\mathbf{E}(\mathbf{x},t) = e^{j[\omega_1 t+\varphi(\mathbf{x},\omega_1)]}\, \boldsymbol{e}(\mathbf{x},\omega_1)$ with the claim that $\boldsymbol{e}(\mathbf{x},\omega_1)$ is real, because you can only do that for one "reference" component.

How might I repair this? This is all I can do:

$$E_i(\mathbf{x},t) = \exp(j[\omega_1 t + \varphi_{Ei}(x,\omega)])\, e_i(\mathbf{x},\omega_1)$$
$$B_i(\mathbf{x},t) = \exp(j[\omega_1 t + \varphi_{Bi}(x,\omega)])\, b_i(\mathbf{x},\omega_1)$$

So let's do a scratch rewrite of Section 1.6 and see what happens. I never really used these phases by the way, except in Appendix D this comes up again!

Resolution: In Section 1.6 I now point out that the form (1.6.5) is NOT the most general form possible, it is just an ansatz form, and in Appendix D we will use a different ansatz form.