Another rewrite of the tail of Section D.9 etc PhL 10.6.14

Things are fine in D.9 above the point I start below. This new stuff was installed 10.6 or 10.7.14 and I think I finally have it right. The trickier the issue, the more words and equations needed to stabilize it.

Lower in this doc are updates to parts of Sections D.11 and D.10. All this is installed.

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How then does $\sigma_d \neq 0$ alter the E field results summarized in box (D.2.33)? The rule is this:

$$N_m \rightarrow N_m' \equiv (\xi_d/\varepsilon_d)\, N_m \quad \text{everywhere .} \tag{D.9.23}$$

Here we use a prime to denote a parameter after the dielectric DC conductivity has been "turned on", and no prime for the case that $G_{dc} = 0$.

But there is another change which must not be overlooked. Although k is treated as a generic constant in Appendix D, we will eventually be setting k to a specific value k(ω) which is determined by activity in the dielectric, which in turn is affected by the dielectric conductance,

$$k = k(\omega) \equiv -j\sqrt{\mathbf{zy}} = -j\sqrt{[R(\omega)+j\omega L(\omega)][G(\omega)+j\omega C(\omega)]}\,, \tag{5.3.6}$$

where the four parameters are as given in the simple model of (Q.1.9). In particular, C(ω) = C, a constant, whereas $G(\omega) = G_{dc} + C \tan_L\omega$, so one may write (since $\tan_L << 1$),

$$k(\omega) = -j\sqrt{[R(\omega)+j\omega L(\omega)][\, G_{dc} + C \tan_L\omega + j\omega C]}$$

$$\approx -j\sqrt{[R(\omega)+j\omega L(\omega)][\, G_{dc}+j\omega C]} \tag{D.9.24}$$

which shows the traditional dependence on G_{dc} and C. The combination $G_{dc} + j\omega C$ may be interpreted in terms of the complex capacitance C' where $G_{dc} + j\omega C = j\omega C'$ as in the line below (1.5.20). Then

$$k(\omega) = -j\sqrt{[R(\omega)+j\omega L(\omega)][\, j\omega C']}\ . \tag{D.9.25}$$

Thus one can write,

$$k' = -j\sqrt{[R(\omega)+j\omega L(\omega)][\, j\omega C']} \qquad // \; G_{dc} > 0 \tag{D.9.26a}$$

$$k = -j\sqrt{[R(\omega)+j\omega L(\omega)][\, j\omega C]}\ . \qquad // \; G_{dc} = 0 \tag{D.9.26b}$$

The point is that the value of k *changes* when G_{dc} is turned on. The k ratio is then

$$\frac{k'}{k} = \sqrt{\frac{C'}{C}} = \sqrt{\xi_d/\varepsilon_d} \tag{D.9.27}$$

where the $\sqrt{\xi_d/\varepsilon_d}$ factor follows from (1.5.19).

Here then is an improved statement of the Rule for how things change when the dielectric conductivity is turned on:

$$N_m \rightarrow N_m' \equiv (\xi_d/\varepsilon_d)\, N_m$$

$$k \rightarrow k' \equiv \sqrt{\xi_d/\varepsilon_d}\; k$$

$$C \rightarrow C' = (\xi_d/\varepsilon_d)\, C \quad . \qquad \text{(D.9.28)}$$

$$\text{where } (\xi_d/\varepsilon_d) = \frac{C'}{C} = \frac{G_{dc} + j\omega C}{j\omega C} = 1 + \frac{(G_{dc}/C)}{j\omega} = 1 + \frac{(\sigma_d/\varepsilon_d)}{j\omega} \qquad \text{// see (4.4.10)}$$

For example, consider (D.1.8) in its form for $G_{dc} = 0$ [recall $I = 2\pi a\, (\omega/k)\, N_0$ from (D.2.31a)],

$$N_0 = \langle n(\theta) \rangle = q/(2\pi a) = CV/(2\pi a) \quad \Rightarrow \quad I = CV\,(\omega/k) \quad . \qquad \textit{(D.1.8)}$$

Here $n = n_s$ and $q = q_s$ which are the surface charge and its cross section integral, so the above really says

$$N_0 = \langle n_s(\theta) \rangle = q_s/(2\pi a) = CV/(2\pi a) \quad \Rightarrow \quad I = CV\,(\omega/k) \quad . \qquad \textit{(D.1.8)}$$

When $G_{dc} > 0$, the surface charges become their "transport charge" alter egos,

$$n_s \rightarrow n' = n_c = (\xi_d/\varepsilon_d) n_s \qquad \text{// (1.5.17)}$$
$$q_s \rightarrow q' = q_c = (\xi_d/\varepsilon_d) q_s \qquad \text{// integral of the above} \qquad \text{(D.9.29)}$$

and then (D.1.8) becomes

$$N_0' = \langle n_c(\theta) \rangle = q_c/(2\pi a) = C'V/(2\pi a) \quad \Rightarrow \quad I' = C'V\,(\omega/k') \qquad \textit{(D.1.8)}$$

or

$$(\xi_d/\varepsilon_d) N_0 = (\xi_d/\varepsilon_d) \langle n_s(\theta) \rangle = (\xi_d/\varepsilon_d)\, q_s/(2\pi a) = (\xi_d/\varepsilon_d)\, C \quad V/(2\pi a)$$

$$\Rightarrow \quad I' = C'V\,(\omega/k') = (\xi_d/\varepsilon_d)\, C\, V \frac{\omega}{\sqrt{\xi_d/\varepsilon_d}\; k} = CV\,(\omega/k) \sqrt{\xi_d/\varepsilon_d} = I \sqrt{\xi_d/\varepsilon_d} \qquad \text{(D.9.30)}$$

and we find that the total current increases by this ratio when G_{dc} is turned on,

$$\frac{I'}{I} = \sqrt{\xi_d/\varepsilon_d} \quad . \qquad \text{(D.9.31)}$$

A more direct path to this conclusion is the following, again using (D.2.31a) that $I = 2\pi a\, (\omega/k)\, N_0$:

$$I = 2\pi a\, (\omega/k)\, N_0$$

$$\rightarrow\ I' = 2\pi a\,(\omega/k')\,N_0' = 2\pi a\,\frac{\omega}{\sqrt{\xi_d/\varepsilon_d}\ k}\,(\xi_d/\varepsilon_d)\,N_0 = \sqrt{\xi_d/\varepsilon_d}\ 2\pi a\,\frac{\omega}{k}\,N_0 = \sqrt{\xi_d/\varepsilon_d}\ \ I\,. \qquad \text{(D.9.32)}$$

As a simple verification of this claim, consider the G_{dc} "turn on" viewed from this perspective,

$$I = V/Z_0 \quad \text{where} \quad 1/Z_0 = \sqrt{\frac{j\omega C}{R + j\omega L}} \qquad \text{(D.9.33a)}$$

$$I' = V/Z_0' \quad \text{where} \quad 1/Z_0' = \sqrt{\frac{G_{dc} + j\omega C}{R + j\omega L}} = \sqrt{\frac{j\omega C'}{R + j\omega L}} \qquad \text{(D.9.33b)}$$

where in general $Z_0 = \sqrt{\mathbf{z}/\mathbf{y}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$ as in (4.12.18). From (D.9.33) we must have

$$\frac{I'}{I} = \sqrt{\frac{j\omega C'}{j\omega C}} = \sqrt{\frac{C'}{C}} = \sqrt{\xi_d/\varepsilon_d}$$

which agrees with (D.9.31) above.

Now consider the Appendix D E fields from (D.2.33) as stated for $G_{dc} = 0$:

Second summary of the E field solutions : $\quad R_{dc} = \frac{1}{\sigma\pi a^2} \quad \beta'^2 = \beta^2 - k^2 \qquad$ *(D.2.33)*

$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\ (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$

$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\ (ak)\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$

$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\ (ak)\ h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$

$I = 2\pi a\,(\omega/k)\,N_0 = C\,V\,(\omega/k) \qquad$ *(D.2.31a)* and *(D.1.8)* $\qquad G_{dc} = 0$

When G_{dc} is turned on, the fields are instead given by [taking I→I' and k→k' and C → C']

Second summary of the E field solutions : $\quad R_{dc} = \frac{1}{\sigma\pi a^2} \quad \beta'^2 = \beta^2 - k'^2 \qquad$ (D.9.34)

$E_z(r,m) = (1/4)\,\eta_m\, I'\, R_{dc}\ (a\beta')\, f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$

$E_r(r,m) = (j/4)\,\eta_m\, I'\, R_{dc}\ (ak')\, g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$

$E_\theta(r,m) = (1/4)\,\eta_m\, I'\, R_{dc}\ (ak')\ h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$

$I' = 2\pi a\,(\omega/k')\,N_0' = C'\,V\,(\omega/k') \qquad$ *(D.2.31a)* and *(D.1.8)* $\qquad G_{dc} > 0$

It is convenient to express everything in terms of k', so we use

$I' = C'\,V\,(\omega/k') = (\xi_d/\varepsilon_d)\,CV\,(\omega/k')$

to rewrite the above box as

Second summary of the E field solutions : $\qquad R_{dc} = \frac{1}{\sigma\pi a^2} \qquad \beta'^2 = \beta^2 - k'^2 \qquad$ (D.9.35)

$E_z(r,m) = (1/4)\,(\xi_d/\varepsilon_d)\,\eta_m\,CV\,R_{dc}\,(\omega/k')\ (a\beta')\,f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$

$E_r(r,m) = (j/4)\,(\xi_d/\varepsilon_d)\,\eta_m\,CV\,R_{dc}\,(\omega a)\,g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$

$E_\theta(r,m) = (1/4)\,(\xi_d/\varepsilon_d)\,\eta_m\,CV\,R_{dc}\,(\omega a)\,h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad G_{dc} > 0$

Then to get back to $G_{dc} = 0$, one replaces $(\xi_d/\varepsilon_d) \rightarrow 1$ and $k' \rightarrow k$. Because k' appears as part of β' and β' appears in x and x_a and these are Bessel function arguments, one *cannot* simply say that the E_i fields are scaled up by the factor (ξ_d/ε_d) when the DC conductivity of the dielectric is turned on. This is the case, however, when $|k'| << |\beta|$ which is the situation for large ω (see (D.2.2) and following text).

Now in our application of the above fields, we will always be using the symbol k *with the understanding* that k = k(ω) = with G_{dc} present or absent as appropriate, In this light we do a final rewrite,

Second summary of the E field solutions : $\qquad R_{dc} = \frac{1}{\sigma\pi a^2} \qquad \beta'^2 = \beta^2 - k^2 \qquad$ (D.9.36)

$E_z(r,m) = (1/4)\,(\xi_d/\varepsilon_d)\,\eta_m\,CV\,R_{dc}\,(\omega/k)\ (a\beta')\,f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$

$E_r(r,m) = (j/4)\,(\xi_d/\varepsilon_d)\,\eta_m\,CV\,R_{dc}\,(\omega a)\,g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$

$E_\theta(r,m) = (1/4)\,(\xi_d/\varepsilon_d)\,\eta_m\,CV\,R_{dc}\,(\omega a)\,h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad G_{dc} > 0$

and now to "get back to $G_{dc} = 0$" we just do $(\xi_d/\varepsilon_d) \rightarrow 1$. In this form, one gets the misleading impression that all E fields are simply scaled up by the factor (ξ_d/ε_d), but this is just an artifact of our notation.

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Now I will install this soon, but I want to jump ahead now Sections D.10 and D.11 where I take large and small ω limits of things.

First D.10: Right off the bat I refer to the E fields in box (D.2.33), and maybe I will change this reference, but let it be for now.

(a) symmetry of f_m and E field expansions. This is all for G = 0 implicitly.

(b) limits of f_m for large ω, and we end up with box (D.10.13) again for G = 0.

Comment: I think I could just add (ξ_d/ε_d) to each E_i to get the G > 0 case. Yes, but no need to do that.

OK, Section D.10 is OK, and I never mention G > 0.

Second D.11. Continue here.

(a) my excuse for things being wrong, no changes here. I do quote the G = 0 App D E fields. I don't want to rock the boat on this little section.

(b) I state k and β' for low ω and G > 0. All is OK. I then state same for low ω and G = 0. All OK.

(c) Obtain small-x versions of f_m etc, all OK.

(d) Now here I should do things for both G = 0 and G>0 cases. I keep things as they were for G = 0 only, then I add a small comment to show that my J_z asymmetry ratio is unaltered for G > 0 !

Things are now OK through the end of my anomaly paragraph.

Reconsider from above for G > 0

Second summary of the E field solutions : $\quad R_{dc} = \dfrac{1}{\sigma\pi a^2} \qquad \beta'^2 = \beta^2 - k^2 \qquad$ (D.9.36)

$$E_z(r,m) = (1/4)\,(\xi_d/\varepsilon_d)\,\eta_m\,CV\,R_{dc}\,(\omega/k)\,(a\beta')\,f_m \qquad f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

$$E_r(r,m) = (j/4)\,(\xi_d/\varepsilon_d)\,\eta_m\,CV\,R_{dc}\,(\omega a)\,g_m \qquad g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a$$

$$E_\theta(r,m) = (1/4)\,(\xi_d/\varepsilon_d)\,\eta_m\,CV\,R_{dc}\,(\omega a)\,h_m \qquad h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad G_{dc} > 0$$

Write this in the low ω limit as

$$E_z(r,m) = (1/4)\,(\xi_d/\varepsilon_d)\,\eta_m\,CV\,R_{dc}\,(\omega/k)\,(a\beta')\,(r/a)^m\,(m+1)\,(2/\beta' a)$$
$$E_r(r,m) = (j/4)\,(\xi_d/\varepsilon_d)\,\eta_m\,CV\,R_{dc}\,(\omega a)\,[(r/a)^{m+1} + (r/a)^{m-1}]$$
$$E_\theta(r,m) = (1/4)\,(\xi_d/\varepsilon_d)\,\eta_m\,CV\,R_{dc}\,(\omega a)\,[(r/a)^{m+1} - (r/a)^{m-1}]$$

$$E_z(r,0) = (1/4)\,(\xi_d/\varepsilon_d)\,CV\,R_{dc}\,(\omega/k)\,(a\beta')\,4/(a\beta')$$
$$E_r(r,0) = (j/4)\,(\xi_d/\varepsilon_d)\,CV\,R_{dc}\,(\omega a)\,2\,(r/a)$$

$E_\theta(r,0) = 0$

and again

$$E_z(r,m) = (1/2)\,(\xi_d/\varepsilon_d)\,\eta_m\,CV\,R_{dc}\,(\omega/k)\,(r/a)^m\,(m+1)$$
$$E_r(r,m) = (j/4)\,(\xi_d/\varepsilon_d)\,\eta_m\,CV\,R_{dc}\,(\omega a)\,[(r/a)^{m+1} + (r/a)^{m-1}]$$
$$E_\theta(r,m) = (1/4)\,(\xi_d/\varepsilon_d)\,\eta_m\,CV\,R_{dc}\,(\omega a)\,[(r/a)^{m+1} - (r/a)^{m-1}]$$

$$E_z(r,0) = (\xi_d/\varepsilon_d)\,CV\,R_{dc}\,(\omega/k)$$
$$E_r(r,0) = (j/4)\,(\xi_d/\varepsilon_d)\,CV\,R_{dc}\,(\omega a)\,2\,(r/a)$$
$$E_\theta(r,0) = 0$$

But for small ω I know that

$$(\xi_d/\varepsilon_d) = \frac{(G_{dc}/C)}{j\omega} = G/(j\omega C)$$

so install this to get

$$E_z(r,m) = (1/2)\,G/(j\omega C)\,\eta_m\,CV\,R_{dc}\,(\omega/k)\,(r/a)^m\,(m+1)$$
$$E_r(r,m) = (j/4)\,G/(j\omega C)\eta_m\,CV\,R_{dc}\,(\omega a)\,[(r/a)^{m+1} + (r/a)^{m-1}]$$
$$E_\theta(r,m) = (1/4)\,G/(j\omega C)\,\eta_m\,CV\,R_{dc}\,(\omega a)\,[(r/a)^{m+1} - (r/a)^{m-1}]$$

$$E_z(r,0) = G/(j\omega C)\,CV\,R_{dc}\,(\omega/k)$$
$$E_r(r,0) = (j/4)\,G/(j\omega C)CV\,R_{dc}\,(\omega a)\,2\,(r/a)$$
$$E_\theta(r,0) = 0$$

or

$$E_z(r,m) = (1/2)\,G/(j)\,\eta_m\,V\,R_{dc}\,(1/k)\,(r/a)^m\,(m+1)$$
$$E_r(r,m) = (j/4)\,G/(j)\,\eta_m\,V\,R_{dc}\,(a)\,[(r/a)^{m+1} + (r/a)^{m-1}]$$
$$E_\theta(r,m) = (1/4)\,G/(j)\,\eta_m\,V\,R_{dc}\,(a)\,[(r/a)^{m+1} - (r/a)^{m-1}]$$

$$E_z(r,0) = G/(j)\,V\,R_{dc}\,(1/k)$$
$$E_r(r,0) = (j/4)\,G/(j)V\,R_{dc}\,(a)\,2\,(r/a)$$
$$E_\theta(r,0) = 0$$

But there is more. We have for $G > 0$ that $k \approx -j\sqrt{R_{dc2}G}$ so then $1/k = j/\sqrt{R_{dc2}G}$. We then get

$$E_z(r,m) = (1/2)\,G/(j)\,\eta_m\,V\,R_{dc}\,(j/\sqrt{R_{dc2}G})\,(r/a)^m\,(m+1)$$
$$E_r(r,m) = (j/4)\,G/(j)\,\eta_m\,V\,R_{dc}\,(a)\,[(r/a)^{m+1} + (r/a)^{m-1}]$$
$$E_\theta(r,m) = (1/4)\,G/(j)\,\eta_m\,V\,R_{dc}\,(a)\,[(r/a)^{m+1} - (r/a)^{m-1}]$$

$E_z(r,0) = G/(j)\ V\ R_{dc}\ (j/\sqrt{R_{dc2}G})$
$E_r(r,0) = (j/4)\ G/(j)V\ R_{dc}\ (a)\ 2\ (r/a)$
$E_\theta(r,0) = 0$

or

$E_z(r,m) = (1/2)\ G\ \eta_m\ V\ R_{dc}\ (1/\sqrt{R_{dc2}G})\ (r/a)^m\ (m+1)$
$E_r(r,m) = (1/4)\ G\ \eta_m\ V\ R_{dc}\ (a)\ [(r/a)^{m+1} + (r/a)^{m-1}]$
$E_\theta(r,m) = -j(1/4)\ G\ \eta_m\ V\ R_{dc}\ (a)\ [(r/a)^{m+1} - (r/a)^{m-1}]$

$E_z(r,0) = G\ V\ R_{dc}\ (1/\sqrt{R_{dc2}G})$
$E_r(r,0) = (1/2)\ G\ V\ R_{dc}\ (r)$
$E_\theta(r,0) = 0$

or

$E_z(r,m) = (1/2)\ \eta_m\ V\ R_{dc}\ \sqrt{G/R_{dc}}\ (r/a)^m\ (m+1)$
$E_r(r,m) = (1/4)\ \eta_m\ V\ R_{dc}\ (G\ a)\ [(r/a)^{m+1} + (r/a)^{m-1}]$
$E_\theta(r,m) = -j(1/4)\ \eta_m\ V\ R_{dc}\ (G\ a)\ [(r/a)^{m+1} - (r/a)^{m-1}]$

$E_z(r,0) = V\ R_{dc}\ \sqrt{G/R_{dc}}$
$E_r(r,0) = (1/2)\ V\ R_{dc}\ (G\ r)$
$E_\theta(r,0) = 0$

Another attempt to rewrite section D.11 (d)

(d) Low frequency E fields

Recall that for the general case $G \geq 0$ we can write the E fields as

Second summary of the E field solutions : $R_{dc} = \frac{1}{\sigma\pi a^2}$ $\beta'^2 = \beta^2 - k^2$ *(D.9.36)*

$E_z(r,m) = (1/4)\ (\xi_d/\varepsilon_d)\ \eta_m\ CV\ R_{dc}\ (\omega/k)\ (a\beta')\ f_m$ $f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right]$ $x = \beta' r$

$E_r(r,m) = (j/4)\ (\xi_d/\varepsilon_d)\ \eta_m\ CV\ R_{dc}\ (\omega a)\ g_m$ $g_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$ $x_a = \beta' a$

$E_\theta(r,m) = (1/4)\ (\xi_d/\varepsilon_d)\ \eta_m\ CV\ R_{dc}\ (\omega a)\ h_m$ $h_m = \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right]$

For small ω, we use the small-x limits of (D.11.6) to get

$E_z(r,m) = (1/2)\ (\xi_d/\varepsilon_d)\ \eta_m\ CV\ R_{dc}\ (\omega/k)\ (r/a)^m\ (m+1)$
$E_r(r,m) = (j/4)\ (\xi_d/\varepsilon_d)\ \eta_m\ CV\ R_{dc}\ (\omega a)\ [(r/a)^{m+1} + (r/a)^{m-1}]$

$$E_\theta(r,m) = (1/4)\,(\xi_d/\varepsilon_d)\,\eta_m\,CV\,R_{dc}\,(\omega a)\,[(r/a)^{m+1} - (r/a)^{m-1}] \qquad m > 0$$

$$E_z(r,0) = (\xi_d/\varepsilon_d)\,CV\,R_{dc}\,(\omega/k) \qquad (D.11.7)$$
$$E_r(r,0) = (j/2)\,(\xi_d/\varepsilon_d)\,CV\,R_{dc}\,(\omega r)$$
$$E_\theta(r,0) = 0 \qquad m = 0$$

For G = 0 one has,

$$(\xi_d/\varepsilon_d) = 1 \qquad \textit{(D.9.28)}$$

$$k = \sqrt{R_{dc2}C}\;\sqrt{\omega}\;e^{-j/4} \quad => \quad C(\omega/k) = e^{j/4}\,\sqrt{\omega}\,\sqrt{C/R_{dc2}} \qquad \textit{(D.11.2)}$$

and then the E fields are

$$E_z(r,m) = (1/2)\,\eta_m\,e^{j/4}\,V\,R_{dc}\,\sqrt{\omega}\,\sqrt{C/R_{dc2}}\,(r/a)^m\,(m+1) \qquad (D.11.8)$$
$$E_r(r,m) = (j/4)\,\eta_m\,CV\,R_{dc}\,(\omega a)\,[(r/a)^{m+1} + (r/a)^{m-1}]$$
$$E_\theta(r,m) = (1/4)\,\eta_m\,CV\,R_{dc}\,(\omega a)\,[(r/a)^{m+1} - (r/a)^{m-1}] \qquad m > 0$$

$$E_z(r,0) = e^{j/4}\,V\,R_{dc}\,\sqrt{\omega}\,\sqrt{C/R_{dc2}}$$
$$E_r(r,0) = (j/2)\,CV\,R_{dc}\,(\omega r)$$
$$E_\theta(r,0) = 0 \qquad m = 0 \qquad G = 0 \qquad \omega \to 0$$

As $\omega \to 0$, all E fields go to zero as we might expect since $Z_0 \to \infty$. In the network model one has,

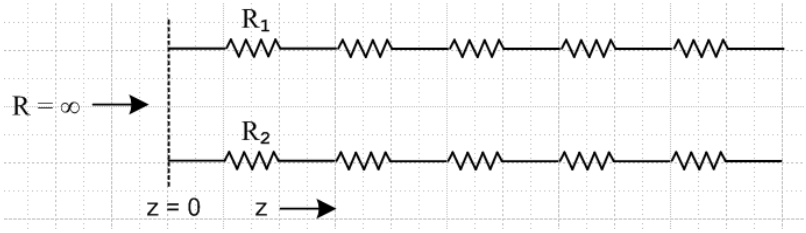

Fig D.8

The current is $I = CV(\omega/k) = V\,e^{j/4}\,\sqrt{\omega}\,\sqrt{C/R_{dc2}}$ and it too $\to 0$. With no current in the transmission line, it seems reasonable that all E fields should vanish.

For G > 0 one has instead, for small ω,

$$(\xi_d/\varepsilon_d) \approx (G/j\omega C) \qquad \textit{(D.9.28)}$$

$$k = -j\sqrt{R_{dc2}G} \qquad \textit{(D.11.1)}$$

and then the E fields of *** are,

$$E_z(r,m) = (1/2)\,\eta_m\,V\,R_{dc}\,\sqrt{G/R_{dc}}\,(r/a)^m\,(m+1) \qquad (D.11.9)$$

$$E_r(r,m) = (1/4)\,\eta_m\,V\,R_{dc}\,(G\,a)\,[(r/a)^{m+1} + (r/a)^{m-1}]$$
$$E_\theta(r,m) = -j(1/4)\,\eta_m\,V\,R_{dc}\,(G\,a)\,[(r/a)^{m+1} - (r/a)^{m-1}] \qquad m > 0$$

$$E_z(r,0) = V\,R_{dc}\,\sqrt{G/R_{dc}}$$
$$E_r(r,0) = (1/2)\,V\,R_{dc}\,(G\,r)$$
$$E_\theta(r,0) = 0 \qquad m = 0 \qquad G > 0 \quad \omega \to 0$$

As $\omega \to 0$, all E fields approach finite values except $E_\theta(r,0) = 0$.

Consider now the ratio $E_z(r,m)/ E_z(r,0)$. For G = 0 or G > 0 the ratio is exactly the same, namely

$$\frac{J_z(r,m)}{J_z(r,0)} = \frac{E_z(r,m)}{E_z(r,0)} = (1/2)\,\eta_m\,(r/a)^m\,(m+1) \qquad \text{as } \omega \to 0,\ m > 0 \qquad \text{(D.11.10)}$$

where $J_z = \sigma E_z$ is used to obtain the leftmost ratio.

An anomaly. For the G = 0 case, the above result is rather disturbing. In the DC limit $\omega \to 0$, we expect the current density J_z inside our round wire to approach a uniform value. This is so because we expect there to be no eddy currents at $\omega = 0$ and these are the cause of J_z non-uniformity as discussed in Appendix P. In order for J_z to be uniform, the partial wave components J(r,m) for $m \neq 0$ must all vanish. But the ratio above shows that they do not vanish in relation to J(r,0). It is true that in the final limit $\omega \to 0$ all E fields vanish, as shown in ***, but we would have expected that as we approach the limit, the current density would smoothly approach a uniform constant value.

Let us re-examine this situation in the θ domain rather than the m domain. Recall that for n(θ) even, we have $\eta_{-m} = \eta_m$ and then

$$E_z(r,\theta) = E_z(r,m=0) + 2\sum_{m=1}^{\infty} E_z(r,m)\cos(m\theta) \quad . \qquad \textit{(D.10.3)}$$

Inserting our small-ω fields from *** then gives

$$E_z(r,\theta) = e^{j/4}\,V\,R_{dc}\,\sqrt{\omega}\,\sqrt{C/R_{dc2}}$$
$$+ 2\sum_{m=1}^{\infty} (1/2)\,\eta_m\,e^{j/4}\,V\,R_{dc}\,\sqrt{\omega}\,\sqrt{C/R_{dc2}}\,(r/a)^m\,(m+1)\cos(m\theta)$$
$$= e^{j/4}\,V\,R_{dc}\,\sqrt{\omega}\,\sqrt{C/R_{dc2}}\,\left[\,1 + \sum_{m=1}^{\infty} \eta_m\,(r/a)^m\,(m+1)\cos(m\theta)\,\right]$$
$$= (1/\sigma)\,J_z(r,\theta) \qquad \text{(D.11.11)}$$

and one sees explicitly now how $J_z(r,\theta)$ is not uniform but varies with both r and θ. As $\omega \to 0$, the shape of J_z over the round wire cross section approaches the bracketed function.

There is a possible obscure argument that somehow, as $\omega \rightarrow 0$, the ratio of the eddy currents to the total current I is somehow constant and that is how the J_z asymmetry is maintained all the way to $\omega \rightarrow 0$. We don't think this argument is valid, and we really do have an anomaly of our theory as $\omega \rightarrow 0$.

Reader Exercise: Does (D.11.9) give the correct solution to the implied magnetostatics problem, or are there anomalies like the one noted above? Notice that Z_0 is certainly correct based on the reader exercise given in Appendix K (c). The "wave" decays in z according to $e^{-jkz} = \exp(-\sqrt{RG}\, z)$ which also seems reasonable.

****

The above two pieces are now installed, but now I have more trouble in Section D.10, so write

$$E_i(r,\theta) = E_i(r,m=0) + 2 \sum_{m=1}^{\infty} E_i(r,m) \cos(m\theta) \qquad \text{(D.10.3)}$$

Then for even n(θ) the E fields are, using (D.9.37) with $\mathcal{B} \equiv (\xi_d/\varepsilon_d)\, CV\, R_{dc}$,

$$E_z(r,\theta) = (1/4)\, \eta_m\, \mathcal{B}\, (\omega a)\, (\beta'/k)\, [\, f_0 + 2\, \Sigma_{m=1}^{\infty}\, f_m\, \eta_m\, \cos(m\theta)\,]$$

$$E_r(r,\theta) = (j/4)\, \eta_m\, \mathcal{B}\, (\omega a)\, [\, g_0 + 2\, \Sigma_{m=1}^{\infty}\, g_m\, \eta_m\, \cos(m\theta)\,]$$

$$E_\theta(r,\theta) = (1/4)\, \eta_m\, \mathcal{B}\, (\omega a)\, [\, h_0 + 2\, \Sigma_{m=1}^{\infty}\, h_m\, \eta_m\, \cos(m\theta)\,] \ . \qquad \text{(D.10.4a)}$$

For general n(θ) where η_m and η_{-m} are no longer equal we have instead

$$E_z(r,\theta) = (1/4)\, \eta_m\, \mathcal{B}\, (\omega a)\, (\beta'/k)\, [\Sigma_{m=-\infty}^{\infty}\, f_m\, \eta_m\, e^{jm\theta}\,]$$

$$E_r(r,\theta) = (j/4)\, \eta_m\, \mathcal{B}\, (\omega a)\, [\Sigma_{m=-\infty}^{\infty}\, g_m\, \eta_m\, e^{jm\theta}\,]$$

$$E_\theta(r,\theta) = 1/4)\, \eta_m\, \mathcal{B}\, (\omega a)\, [\Sigma_{m=-\infty}^{\infty}\, h_m\, \eta_m\, e^{jm\theta}\,] \ . \qquad \text{(D.10.4b)}$$

Then things are OK for a while, but then above (D.10.13) I need to fix again

The E field expressions from box (D.9.37) are then,

$$E_z(r,m) = (1/4)\, \eta_m\, \mathcal{B}\, (\omega a)\, (\beta'/k)\, f_m \qquad f_m = -2j \sqrt{\frac{a}{r}}\, e^{(1+j)(r-a)/\delta} \qquad x = \beta' r$$

$$E_r(r,m) = (j/4)\, \eta_m\, \mathcal{B}\, (\omega a)\, g_m \qquad g_m = 2 \sqrt{\frac{a}{r}}\, e^{(1+j)(r-a)/\delta} \qquad x_a = \beta' a$$

$$E_\theta(r,m) = (1/4)\, \eta_m\, \mathcal{B}\, (\omega a)\, h_m\, h_m \qquad h_m = 0 \qquad k = \beta_{d0}$$

or

Large ω limits of the E field solutions : $R_{dc} = \dfrac{1}{\sigma\pi a^2}$ (D.10.13)

$E_z(r,m) = -(j/2)\,\eta_m\,\mathcal{B}(\omega a)\,(\beta/\beta_{d0})\,\sqrt{\dfrac{a}{r}}\,e^{(1+j)(r-a)/\delta}$ $\qquad x = \beta r$

$E_r(r,m) = (j/2)\,\eta_m\,\mathcal{B}(\omega a)\sqrt{\dfrac{a}{r}}\,e^{(1+j)(r-a)/\delta}$ $\qquad x_a = \beta a$

$E_\theta(r,m) = 0$

********** still more!!! **********

The θ-space fields from (D.10.4b) are then,

$E_z(r,\theta) = -(j/2)\,\mathcal{B}(\omega a)\,(\beta/k)\,\{\sqrt{\dfrac{a}{r}}\,e^{(1+j)(r-a)/\delta}\}\,[\,\Sigma_{m=-\infty}^{\infty}\,\eta_m\,e^{jm\theta}\,]$

$E_r(r,\theta) = (j/2)\,\mathcal{B}(\omega a)\,\{\sqrt{\dfrac{a}{r}}\,e^{(1+j)(r-a)/\delta}\}\,[\Sigma_{m=-\infty}^{\infty}\,\eta_m\,e^{jm\theta}\,]$

$E_\theta(r,\theta) = 0$ $\qquad \eta_m = N_m/N_0$. (D.10.14)

But the [...] expansions shown here are just $n(\theta)/N_0$ from (D.1.5a). Meanwhile,

$(j/2)\,\mathcal{B}(\omega a)/N_0 = (j/2)\,(\xi_d/\varepsilon_d)\,CV\,R_{dc}\,(\omega a)\,/\,N_0$

$= (j/2)\,(1)\,[\,2\pi a\,N_0]\,R_{dc}\,(\omega a)\,/\,N_0$ // (D.1.8) and $\xi_d/\varepsilon_d = 1$ for large ω

$= (j/2)\,2\pi a\,(1/\sigma\pi a^2)\,(\omega a)$ // $R_{dc} = 1/\sigma\pi a^2$

$= (j\omega/\sigma)$

so then for large ω (D.10.14) becomes,

****
so both E_z and E_r track with $n(\theta)$. From (D.10.14) the surface impedance is then

$Z_s(\theta) \equiv E_z(a,\theta)\,/\,I$

$E_z(r,\theta) = -\,(j\omega/\sigma)\,n(\theta)\,(\beta/k)$

$I = 2\pi a\,(\omega/k)\,N_0$ $\qquad$ large ω

$Z_s(\theta) \equiv (j\omega/\sigma)\,n(\theta)\,(\beta/k)\,/\,2\pi a\,(\omega/k)\,N_0$

$$= \frac{n(\theta)}{\langle n(\theta) \rangle} \quad (j/\sigma)\ (\beta/2\pi a)$$

$$= \frac{n(\theta)}{\langle n(\theta) \rangle} \quad (j/\sigma)\ (\beta/2\pi a)$$

From (2.4.7) we know $\beta = (j-1)/\delta$, so

$$Z_{\mathbf{s}}(\theta) = \frac{n(\theta)}{\langle n(\theta) \rangle} \quad (j/\sigma)\ (1/2\pi a\delta)\ (j-1) = -\ \frac{1}{\sigma(2\pi a)\delta}\ (1+j)$$

****

Uncle!! I have tried iteratively to directly update D.9, D.10 and D.11 in lines doc. I think this sections are now done, but cross references will need fixing and we need to do Section 6.5 and Chapter 7 updates.