Note that page numbering is turned on in this template.

Appendix F: Waveguides

F.1 Discussion

A transmission line must have at least two distinct conductors to carry the TEM wave described in Section 3.7 and as illustrated in the figures there. For a two conductor transmission line the surfaces of the conductors have a potential difference of amplitude $V \neq 0$.

A single wire cannot carry a TEM wave except in the sense of Section 2.1 where it acts as the center conductor of a coaxial cable with a far-distant return sheath. A TEM wave cannot propagate down the inside of a hollow pipe regardless of cross section shape since the continuous conductor cross section "shorts out" any possible $V \neq 0$.

In this document we have associated the TEM wave with the phrase "transmission line". but certainly a waveguide is a form of transmission line. Normally one associates the word "waveguide" with the TE and TM modes such waveguides carry. The usual form of a waveguide is in fact a hollow pipe, often of rectangular or circular cross section. However, it is possible for a 2 conductor transmission line to have TE and TM modes. In this Appendix we shall not present a theory of waveguides since that is well done in Jackson and many other texts, but we would like to show that a transmission line made from two closely spaced parallel plates can carry waveguide modes in addition to the TEM mode. And we want to use this simple example to illustrate the notion that waveguide modes have lower cutoff frequencies whereas the TEM mode can operate all the way down to $\omega = 0$.

The terminology TEM (Transverse Electric and Magnetic) means that both the E and B fields are transverse, as shown in Figures 3.5 through 3.7. In reality, we know there is a very small longitudinal E_z field because E_z is continuous at a conductor surface and we know $J_z = \sigma E_z$ just inside the conductor. This E_z field exists and has a cosine-like shape between the conductors, having the opposite direction at the second conductor. This field might be smaller than the transverse E field by a factor 10^{-4} as shown in (3.6.1).

A TEM wave is very much like a plane wave with its transverse **E** and **B** fields, but the fields are distorted by the presence of the conductors. As Fig 3.5 shows, this distortion is such that the Poynting vector **E** x **B** always points down the line (z direction), **E** and **B** are always perpendicular at any point, but the **E** field lands perpendicularly on the conductors. The TE and TM modes have much more complicated field patterns.

The waveguide modes are called TE (Transverse Electric) and TM (Transverse Magnetic). The nomenclature is a little confusing since both TE and TM waves generally have transverse E and B fields. The distinction is that the TE modes have no E_z field, while the TM modes have no B_z field. So TE means the E field is "transverse only".

F.2 The TE waveguide modes for a parallel-plate transmission line

We shall assume (an "ansatz") that the entire **E** field is given by

$$\mathbf{E}(x,y,z) = E_y(x)\, e^{j(\omega t - kz)}\ \hat{\mathbf{y}} \qquad \text{(F.2.1)}$$

where we have our usual overloading of the symbol E.

This field in the dielectric must satisfy the ω-domain wave equation (1.5.32) which says

$$(\nabla^2 + \beta^2)\,\mathbf{E} = 0\ . \qquad \text{(F.2.2)}$$

Here β is the usual Helmholtz parameter of the dielectric as in (1.5.1),

$$\beta^2 = \mu\varepsilon\omega^2 - j\omega\mu\sigma = \omega^2\mu\,(\varepsilon - j\sigma/\omega) = \omega^2\mu\,\xi \qquad \xi \equiv \varepsilon - j\sigma/\omega \qquad \textit{(1.5.1)}$$

but in this Appendix we assume the dielectric is non-conducting so $\beta^2 = \omega^2\mu\varepsilon$.

Inserting our ansatz form (F.2.1) for **E** into (F.2.2) we find that

$$(\partial_x^2 + \partial_y^2 + \partial_z^2 + \beta^2)\, E_y(x)\, e^{j(\omega t - kz)} = 0$$

or

$$(\partial_x^2 + 0 + (-k^2) + \beta^2)\, E_y(x) = 0\ . \qquad \text{(F.2.3)}$$

Define

$$\gamma^2 \equiv \beta^2 - k^2 \qquad \text{(F.2.4)}$$

so that

$$(\partial_x^2 + \gamma^2)\, E_y(x) = 0 \qquad \text{(F.2.5)}$$

so

$$E_y(x) = A\sin(\gamma x) + B\cos(\gamma x)\ . \qquad \text{(F.2.6)}$$

We now introduce our parallel plate transmission line (the gap is exaggerated in width)

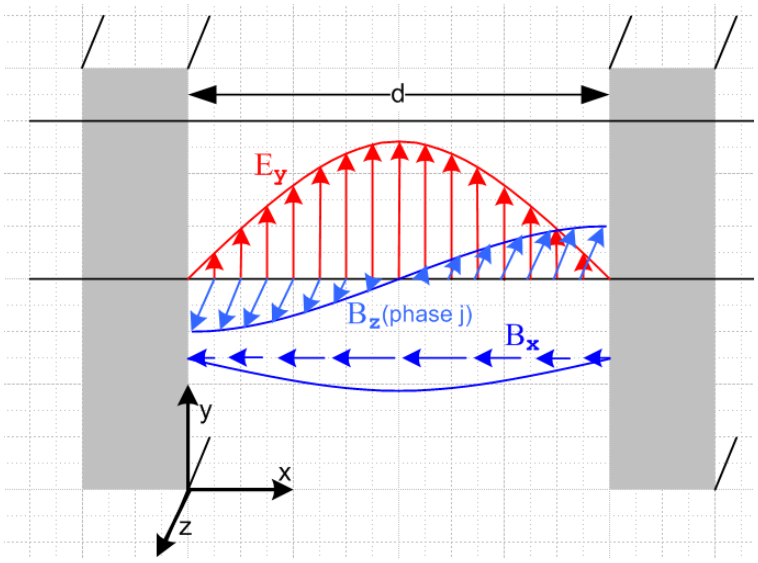

Fig F.1

Since we require $E_y = 0$ at the two inner plate surfaces, we find that

$$E_y(x) = A \sin(\gamma x) \qquad \text{where } \sin(\gamma d) = 0 \qquad \Rightarrow \qquad \gamma d = m\pi \ . \tag{F.2.7}$$

Thus, the parameter γ is quantized by the boundary conditions and we have

$$E_y^{(m)}(x) = A \sin(\gamma_m x) \qquad \gamma_m = m(\pi/d) \qquad m = 1,2,3.... \tag{F.2.8}$$

Suddenly we have "modes" labeled by m. The lowest non-vanishing mode has m = 1, and this is the mode shown in the figure.

We find that the wave's wavenumber k is also quantized. From (F.2.4) we get

$$k_m = \sqrt{\beta^2 - \gamma_m^2} \ . \tag{F.2.9}$$

In order for there to be a traveling wave $e^{j(\omega t - kz)}$, we need k in (F.2.1) to be real, which requires that

$$\beta \geq \gamma_m \ . \tag{F.2.10}$$

For a non-conducting dielectric one has $\beta = \omega\sqrt{\mu\varepsilon} = \omega/v$ where v is the light speeed in the dielectric. Recall from (1.1.29) that $\sqrt{\mu_0\varepsilon_0} = 1/c$. So the above condition is

$$\omega/v \geq m(\pi/d) \qquad \Rightarrow \qquad \omega \geq m(\pi/d)v$$

so

$$\omega \geq \omega_m \qquad \omega_m \equiv m(\pi/d)v = \gamma_m v \ . \tag{F.2.11}$$

Thus the mth TE mode can only operate for ω *above* ω_m, and as m increases the low end mode cutoff increases. For $\omega < \omega_1$ there can be no TE action on this waveguide.

The B fields for our TE mode can be obtained from the Maxwell curl E (1.6.19),

$$\mathbf{B} = (j/\omega)\,\text{curl}\,\mathbf{E} = (j/\omega)\,[\hat{\mathbf{x}}\,(\partial_y E_z - \partial_z E_y) + \hat{\mathbf{y}}\,(\partial_z E_x - \partial_x E_z) + \hat{\mathbf{z}}\,(\partial_x E_y - \partial_y E_x)]$$

$$= (j/\omega)\,[\hat{\mathbf{x}}\,(-\partial_z E_y) + \hat{\mathbf{z}}\,(\partial_x E_y)\,]$$

so then

$$B_x^{(m)}(x) = (j/\omega)(jk)E_y(x) = -(k_m/\omega)\,A\sin(\gamma_m x) \qquad \text{(F.2.12)}$$

$$B_z^{(m)}(x) = (j/\omega)\partial_x E_y(x) = (j/\omega)\,\gamma_m\,A\cos(\gamma_m x)\,. \qquad \text{(F.2.13)}$$

$$E_y^{(m)}(x) = A\sin(\gamma_m x) \qquad \textit{(F.2.8)}$$

If A is real, then E_y and B_x are real and in time phase, while B_z is 90^0 out of phase. An attempt has been made to display all three field components in Fig F.1.

To show that the waveguide mode outlined above is viable, we verify Maxwell's equations. Since the Maxwell curl E equation was used to obtain B, we need verify only the remaining three equations:

$$\text{div}\,\mathbf{E} = \partial_x E_x + \partial_y E_y + \partial_z E_z = \partial_y E_y(x) = 0 \qquad \text{(F.2.14)}$$

$$\text{div}\,\mathbf{B} = \partial_x B_x + \partial_y B_y + \partial_z B_z = \partial_x B_x + \partial_z B_z$$

$$= -(k_m/\omega)\,\gamma_m\,A\cos(\gamma_m x) - jk_m\,(j/\omega)\,\gamma_m\,A\cos(\gamma_m x)$$

$$= -(k_m/\omega)\,\gamma_m\,A\cos(\gamma_m x) + k_m\,(1/\omega)\,\gamma_m\,A\cos(\gamma_m x) = 0 \qquad \text{(F.2.15)}$$

Finally,

$$\text{curl}\,\mathbf{B} = \hat{\mathbf{x}}\,(\partial_y B_z - \partial_z B_y) + \hat{\mathbf{y}}\,(\partial_z B_x - \partial_x B_z) + \hat{\mathbf{z}}\,(\partial_x B_y - \partial_y B_x)$$

$$= + \hat{\mathbf{y}}\,(\partial_z B_x - \partial_x B_z) = \hat{\mathbf{y}}\,[\,(-jk_m) - (k_m/\omega)\,A\sin(\gamma_m x) + (j/\omega)\,\gamma_m^2\,A\sin(\gamma_m x)\,]$$

$$= \hat{\mathbf{y}}\,[\,(jk_m^2/\omega)\,A\sin(\gamma_m x) + (j/\omega)\,\gamma_m^2\,A\sin(\gamma_m x)\,]$$

$$= \hat{\mathbf{y}}\,[\,k_m^2 + \gamma_m^2\,]\,j\,(A/\omega)\sin(\gamma_m x) = \hat{\mathbf{y}}\,\beta^2\,j\,(A/\omega)\sin(\gamma_m x)\,. \qquad \text{(F.2.16)}$$

According to (1.6.18)

$$\text{curl}\,\mathbf{H}(\mathbf{x},\omega) = j\omega\mathbf{D}(\mathbf{x},\omega) + \mathbf{J}(\mathbf{x},\omega) \qquad \textit{(1.6.18)}$$

with $\mathbf{J} = 0$, $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{H} = \mathbf{B}/\mu$ we should have

$$\text{curl } \mathbf{B} = j\omega\varepsilon\mu\, \mathbf{E} \qquad \text{(F.2.17)}$$

or

$$\beta^2\, j\, (A/\omega)\sin(\gamma_m x) = j\omega\varepsilon\mu\, A \sin(\gamma_m x)$$

or

$$\beta^2\, (1/\omega) = \omega\varepsilon\mu$$

or

$$\beta^2 = \omega^2\varepsilon\mu$$

which is (1.5.1) quoted above. Thus we have shown that our TE waveguide modes satisfy all four of Maxwell's equations.

Although the TE and TEM modes are both "transverse electric", there is a significant difference in the E field pattern. In TEM the E field lines run from one conductor to the other so that the line integral of E generates the potential difference V, as shown in Fig 3.5. The E field lines are "sourced by" (or "create") the surface charge on the conductors. In the TE mode of Fig F.1, the E field is still transverse but is parallel to the conductors so the line integral of E between the conductors gives V = 0. These E lines are not sourced by charges on the conductors but are more like the E field lines in a free-space light wave.

Comments:

1. The parallel plate transmission line also has TM waveguide modes, and the cutoff frequencies are the same as for the TE mode.

2. A rectangular waveguide mode has two quantized integers and the cutoff frequency is then a function of both these integers. For TM the E_z field will have sine behavior in both x and y directions.

3. The obvious boundary condition is that $E_t = 0$ at the walls, while a less obvious condition is that $B_n = 0$ at the walls. Notice in our example that $B_x^{(m)}(x) = 0$ at the walls and this is a normal B field.

4. Waveguide problems are normally dealt with using the Helmholtz equation for the E and B fields, whereas the TEM transmission line problem is more easily dealt with using potentials φ and A_z.

5. We have dealt above with an ideal waveguides. In real waveguides the fields E and B penetrate distance δ (skin depth) into the walls and generate ohmic losses causing the wave to be damped. The same thing of course also happens for the transmission line TEM mode.

Reader Exercise: Make a 3D vector plot of the **E** and **B** fields for Fig F.1, and also plot the Poynting vector **S** = **E** x **B** and compare with the TEM wave pattern. Except at the center, in addition to S_z there seems to be an S_x component suggesting a transverse power flow distribution in addition to the expected longitudinal power flow.