Retired old Appendix F PhL 1.15.14

Appendix F: Waveguides

This needs a rewrite or should be thrown out! Most of it seems irrelevant, linkage to the main body of text is weak, not clear what its purpose is, but I guess waveguides are worth mentioning and TE,TM and TEM, but lets get rid of the junk here! I guess I should end up with a few facts which compare the two things. I do think the ideal transmission line has TE and TM at the same time.

Normally a transmission line has *two* conductors and a waveguide has only *one*, but there is no reason why "waveguide action" cannot take place on a transmission line. The following is a very cursory discussion with a simple example.

F.1 A waveguide solution

The standard assumptions made about a "perfect conductor" are that the fields vanish inside, and that all action takes place at the surface, which is basically a mirror. At the surface, tangential **E** fields and normal **B** fields vanish. The tangential **E** field vanishes because the surface is an equipotential, and the normal **B** field vanishes because the Maxwell curl **E** equation (1.1.2) says $\partial_y E_z - \partial_z E_y = -j\omega B_x$ where we assume a point on a metal surface with x being the normal direction. Since E_z and E_y are constantly 0 on the surface, one has $\partial_y E_z = 0$ and $\partial_z E_y = 0$ so $B_x = 0$. For a waveguide, these facts become convenient mathematical boundary conditions, so the *field* wave equations are usually used instead of the *potential* wave equations. There is no ρ or J *inside* a waveguide, so (1.6.2) says,

$$(\nabla^2 + \beta^2)\,\mathbf{E} = 0\ . \qquad (1.6.2) \qquad \text{(F.1.1)}$$

Traditional wave motion $e^{j(\omega t - kz)}$ has phase fronts described by ωt-kz = constant. In time δt the phase front moves distance δz and v = δz/δt is the phase front velocity. Snce ωδt = kδz, we get v = δz/δt = ω/k. For an electromagnetic plane wave traveling in an infinite medium having parameters μ,ε and σ, the value of k is equal to β where

$$\beta^2 \equiv \omega^2 \mu\, \xi \qquad \beta = \text{complex wavenumber}$$
$$\xi \equiv \varepsilon - j\sigma/\omega\ . \qquad \xi = \text{complex dielectric constant} \qquad (1.5.1)$$

We assume that the interior of our waveguide (normally just air) has these parameters μ,ε and σ. For σ ≠ 0, ν and β have small imaginary parts.

In analogy with (1.1.29) that speed of light $c = 1/\sqrt{\mu_0 \varepsilon_0}$ we define a complex phase velocity in the dielectric as

$$v \equiv 1/\sqrt{\mu\xi} \qquad \text{(F.1.2)}$$

and then

$$\beta = \omega/v \qquad \text{(F.1.3)}$$

just as we had k = ω/v in the traditional wave discussion above. If σ is very small then $\xi \approx \varepsilon$ with a very small imaginary part, and v is also mostly real with a small imaginary part.

The first step is to *try* to find a solution to the waveguide problem by separation of variables. If a solution is found, then this separation is justified. Starting back in the time domain, we then try this wave form of solution (wavenumber k),

$$\mathbf{E}(x,y,z,t) = \mathbf{E}(x,y)\, e^{j(\omega t \pm kz)} \quad . \tag{F.1.4}$$

The - sign is for waves traveling in the +z direction. Eq. (F.1.1) then becomes,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2 \right) \mathbf{E}(x,y) = 0 \tag{F.1.5}$$

where

$$\gamma^2 = \beta^2 - k^2 = \frac{\omega^2}{v^2} - k^2 \quad . \tag{F.1.6}$$

As a simple but illustrative example, consider a waveguide consisting of only two plates separated by distance a (in coordinate x). We look for a solution with $\mathbf{E}(x,y) = E_{\mathbf{y}}(x)\, \hat{\mathbf{y}}$ so we have

$$\left(\frac{\partial^2}{\partial x^2} + \gamma^2 \right) E_{\mathbf{y}}(x) = 0 \quad . \tag{F.1.7}$$

A candidate solution is $E_{\mathbf{y}}(x) = A \sin(\gamma x)$. To meet the boundary conditions that $E_{\mathbf{y}}$ vanish at x=0 and x=a, we are forced to set

$$\gamma = \gamma_{\mathbf{m}} \equiv (m\pi/a) \quad . \tag{F.1.8}$$

One says that (F.1.7) and its boundary conditions comprise an eigenvalue problem, and $\gamma_{\mathbf{m}}$ are the eigenvalues.

Here is a sketch of the waveguide E field at some z = constant slice for the m = 1 mode:

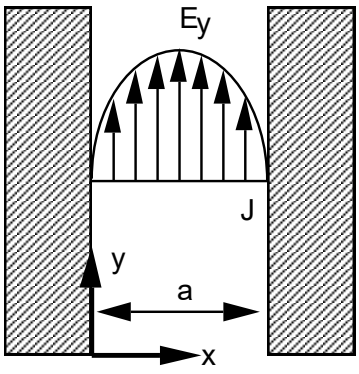


Fig 1: Cross sectional view of a simple waveguide.

If we solve (F.1.6) for the wavenumber k,

$$k = (1/v)\sqrt{\omega^2 - (v\,\gamma_m)^2} \qquad \lambda = 2\pi/k \qquad \omega_m \equiv v\,\gamma_m \text{ = cutoff frequency} \tag{F.1.9}$$

we see clearly that ω must be larger than $\omega_m \equiv v\,\gamma_m$ in order to get a predominantly real k for mode m. Below this ω, k is imaginary and there is no propagation as in (F.1.4), only attenuation. Thus, each mode m has a cutoff frequency ω_m below which there can be no wave. Above the m^{th} cutoff frequency for any ω there is a corresponding real k, so ω and k take on a continuum of legal values. For $\omega_1 < \omega < \omega_2$, for example, only the m = 1 mode can operate.

If the dielectric conductivity σ appearing in ξ is small but non-zero, then even above cutoff the k in (F.1.9) will have some small imaginary part since this is true of v. This results in a slight exponential decay in (F.1.4) caused by heating of the dielectric as the wave moves down the guide.

The other loss mechanism, and a more important one, occurs at the "mirror" surface. Since the walls are not really perfect conductors, the **E** and **B** fields do in fact penetrate some distance (the skin depth), and currents are created according to $\mathbf{J} = \sigma\mathbf{E}$ inside the conductor, so there are heat losses at the walls as well.

These same loss mechanisms occur in transmission lines as well.

The above discussion concerns a particular geometric example of a set of TE waveguide modes m where each mode has a certain cutoff frequency ω_m below which the mode cannot operate. In a TE mode, $E_z = 0$ and the **E** field is entirely tranverse, hence "TE" (in our example, $\mathbf{E} = E_y\hat{\mathbf{y}}$). There exists another set of TM modes (transverse magnetic) in which $B_z = 0$ and the **B** field is entirely transverse. The TM modes have the same cutoff frequencies as the TE modes. See for example Jackson Chapter 8 or Haus and Melcher Chapter 13.3. The transmission line carries a completely different type of wave which is a "TEM" wave in which neither the **E** nor the **B** field has a significant z component. Both fields are essentially transverse.

Note: Both TE and TM modes have transverse electric field components. It's just that in a TE mode the electric field has no longitudinal component in addition to its transverse component.

F.2 A waveguide interpretation

In the previous section, we obtained this approximate solution for the electric field inside the two-plate waveguide,

$$E_y(x,z) = A \sin(\gamma_m x)\, e^{-ikz} \quad \text{with } k^2 = \beta^2 - (\gamma_m)^2 \ . \tag{F.2.1}$$

The solution was approximate because we ignored skin depth effects. It is convenient to think of (F.2.1) as the superposition of two "free space" plane waves of wavenumber β traveling at some skew angle ±θ relative to the z direction, reflecting back and forth off the sides of the waveguide:

$$E_y(x,z) = (Aj/2)\,[\,\exp(-j\boldsymbol{\beta}_1 \bullet \mathbf{r}) - \exp(-j\boldsymbol{\beta}_2 \bullet \mathbf{r})\,] \;= \text{sum of two plane waves} \qquad \text{(F.2.2)}$$

where

$$\boldsymbol{\beta}_1 = \beta_x\,\hat{\mathbf{x}} + \beta_z\,\hat{\mathbf{z}} \quad = \text{wavevector of first wave}$$
$$\boldsymbol{\beta}_2 = -\beta_x\,\hat{\mathbf{x}} + \beta_z\,\hat{\mathbf{z}} \quad = \text{wavevector of second wave} \qquad \text{(F.2.3)}$$

$$(\beta_1)^2 = (\beta_2)^2 = \beta_x^2 + \beta_z^2 = \beta^2$$

$$\tan\theta = \beta_x / \beta_z \;. \qquad \text{(F.2.4)}$$

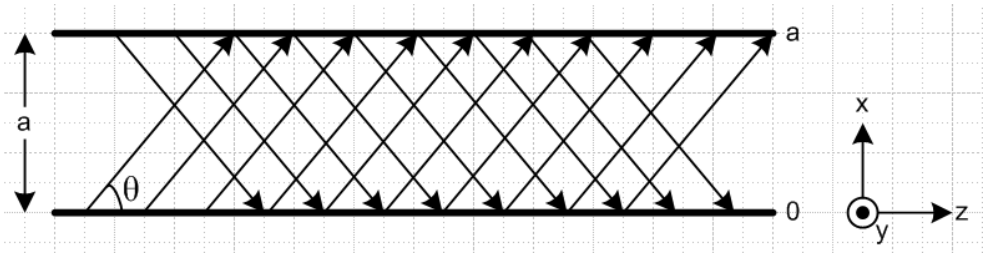
Adding the two terms in (F.2.2) gives

$$E_y(x,z) = A\sin(\beta_x x)\exp(-j\,\beta_z z) \;. \qquad \text{(F.2.5)}$$

Comparing with (F.2.1), we conclude that

$$\beta_x = \gamma_m = m\pi/a \qquad \beta_z = k \qquad \tan\theta = \gamma_m/k = \frac{m\pi v}{a\sqrt{\omega^2 - (v\,\gamma_m)^2}} \;. \qquad \text{(F.2.6)}$$

The two plane waves have wavenumber $\beta = \omega/v$ and not k. Their sum is the superposed wave going down the guide in the z direction with wavenumber k shown in (F.1.4). As ω approaches cutoff from above, $\tan\theta \to \infty$ and $\theta \to \pi/2$ and at cutoff the plane waves just bounce back and forth sideways and there is no propagation down the guide at all. Here is a picture:



The second point to be made involves wavelength. The plane waves have wavelength

$$\lambda = 2\pi/\beta = 2\pi v/\omega.$$

Suppose we try to form a waveguide solution by jamming an integral number of half waves of the plane wave between our two plates at right angles, knowing that in this way we meet the boundary conditions at the two plates:

$$m(\lambda/2) = a \;.$$

Using the above expression for λ gives $\omega = \gamma_m v$, which is the cutoff frequency for mode m in (F.1.9). This is exactly the situation described above when $\theta = 90°$ in (F.2.6). As we move above cutoff in ω, the angle θ decreases from 90, the wavelength λ gets shorter, so the wavenumber β gets *larger* (number of radians of wave per meter). The boundary conditions are maintained by keeping the product of β and $\cos\theta$ constant:

$$\sin(\beta_x a) = 0 \quad \Rightarrow \quad \beta_x a = m\pi \quad \Rightarrow \quad (\beta \cos\theta)a = m\pi \quad \Rightarrow$$

$$\beta\cos\theta = m\pi/a = \gamma_m \quad \Rightarrow \quad \cos\theta = (\gamma_m/\beta) = (\gamma_m v)/\omega = \omega_m/\omega \; .$$

There are several major points that the above discussion is intended to convey. These concern the waveguide modes TE and TM (similar to TE), and not the TEM mode which is what a transmission line does. One should compare the following facts one for one to the corresponding facts which appear in Section 3.8 for the transmission line TEM mode.

Fact 1: In waveguide modes, there can be large tangential transverse currents (J_y). That is, such currents can be large relative to longitudinal currents in the conductor. [I don't think this is a distinguishing factor for waveguides. Transmission lines have losses due to E_z in the conductors. In my example it happens that the losses are due to E_y instead of E_z.Remove this fact!]

Fact 2: The lowest cutoff frequency of a parallel plate waveguide can be obtained by setting the distance between the plates equal to half a wavelength, $a = \lambda/2$. In general, for an arbitrary waveguide, the cutoff occurs when $\lambda/2$ exceeds some similar characteristic transverse dimension of the guide. For ω below cutoff, there can be no waveguide propagation.

Fact 3: In solving waveguide problems, one uses wave equations for the *fields* since the boundary conditions are expressed in terms of fields.