Appendix M Discarded Tail PhL 1.15.14

This seems related to the exterior solution for A_z , notice the $H^{(1)}$ below. File Appendix G.

The remaining

while for the r>a region,

$$A_z^H(r) = [I\mu_2/(\pi a^2)](j/4)2\pi\ (1/\beta_1)\ \{H_0^{(1)}(\beta_1 r)\ aJ_1(\beta_1 a)\} \qquad r > a$$

Since we assumed a uniform current density in (M.5.2), we expect the result to be valid only for small ω and this small β_1^2. One could then at this point use small-argument approximations

It is possible to compute this double integral in the order shown, but this general ω result is not really meaningful because we have assumed a uniform J in the wire, but we know from Chapter 2 that the skin effect is going to occur and J is not uniform. The solution here is meaningful only for small ω where there is no skin effect. As will be shown below, the leading small-x term in $H_0^{(1)}(x)$ is this

$$H_0^{(1)}(x) \approx (2j/\pi)\ln(x)$$

and when this is inserted into (M.5.2) we get

$$A_z^H(r) \approx [I\mu_2/(\pi a^2)]\ (j/4)\ (2j/\pi)\ \int_0^a r'\,dr'\ \int_{-\pi}^{\pi} d\theta'\ \ln[\beta_1 R]$$

$$= [I\mu_2/(\pi a^2)]\ (-1/2\pi)\ (1/2)\ \int_0^a r'\,dr'\ \int_{-\pi}^{\pi} d\theta'\ \ln[\beta_1^2 R^2]$$

$$= -\ (1/4\pi)\ [I\mu_2/(\pi a^2)]\ \int_0^a r'\,dr'\ \int_{-\pi}^{\pi} d\theta'\ \ln[\beta_1^2 R^2]$$

$$= -\ (1/4\pi)\ [I\mu_2/(\pi a^2)]\ \int_0^a r'\,dr'\ \int_{-\pi}^{\pi} d\theta'\ \ln[R^2] \quad + \text{constant}$$

which then replicates our Poisson result (M.4.2) for ω = 0. So we need more terms to get something interesting for low frequency behavior. We need to know more terms of small x limit of $H_0^{(1)}(x)$ in order to use it in our A_z expression above. First, we know that GR7 p 911 that

8.407

$$1.^8 \qquad K_\nu(z) = \frac{\pi i}{2} e^{\frac{\pi}{2}\nu i} H_\nu^{(1)}\left(z e^{\frac{1}{2}\pi i}\right) \qquad \left[-\pi < \arg z \le \tfrac{1}{2}\pi\right]$$

which says

$K_0(z) = (j\pi/2)H_0^{(1)}(jz)$ // same as Jackson p 116 (3.101)

$\Rightarrow \quad K_0(-jx) = (j\pi/2)H_0^{(1)}(x)$ $\quad z = -jx$

so

$H_0^{(1)}(x) = (2/j\pi)\, K_0(-jx)$.

Near z = 0 we have this expansion for $K_0(z)$ from Spiegel 24.40

$K_0(z) = -\,[\ln(z/2) + \gamma\,]I_0(z) + Ax^2 + Bx^4 + \ldots.$ // Spiegel 24.40

Since ln(z/2) is very large near z = 0, we neglect all the non-log terms to get

$K_0(z) \approx -\ln(z/2)I_0(z) \approx -\ln(z/2)\,[\,1 + (1/4)z^2 + (1/64)z^4 + \ldots]$ // Spiegel 24.35

Then for small x we find then that

$H_0^{(1)}(x) \approx -(2/j\pi)\ln(-jx/2)\,[\,1 + (1/4)(-jx)^2 + (1/64)(-jx)^4 + \ldots]$

$\approx +(2j/\pi)\ln(x)\,[\,1 - (1/4)x^2 + (1/64)x^4 + \ldots\,]$

where the ln(-j/2) terms do not get large near x = 0 and so we throw them out as well.

We have seen already how the leading term replicates the ω= 0 results. The next term will be this

$$\Delta A_z^H(r) = [I\mu_2/(\pi a^2)]\,(j/4)\,(2j/\pi)\int_0^a r'\,dr'\int_{-\pi}^{\pi} d\theta'\,\ln[\beta_1 R]\,\{-(1/4)(\beta_1 R)^2\}$$

$$= -(\beta_1^2/4)\,[I\mu_2/(\pi a^2)]\,(j/4)\,(2j/\pi)\int_0^a r'\,dr'\int_{-\pi}^{\pi} d\theta'\,\ln[\beta_1 R]\,R^2$$

$$= -(\beta_1^2/4)\,[I\mu_2/(\pi a^2)]\,(j/4)\,(2j/\pi)(1/2)\int_0^a r'\,dr'\int_{-\pi}^{\pi} d\theta'\,\ln[\beta_1^2 R^2]\,R^2$$

$$= -(\beta_1^2/4)\,[I\mu_2/(\pi a^2)]\,(j/4)\,(2j/\pi)(1/2)\int_0^a r'\,dr'\int_{-\pi}^{\pi} d\theta'\,\ln[R^2]\,R^2 \quad + \text{constant}$$

Comment: This next term isn't very simple! My general solution almost seems simpler!

The enterprising reader can find some way to do this double integral and thereby obtain $A_z^H(r)$, perhaps assume small β_1 . As in the ω = 0 case, we expect that some homogeneous solutions of $(\nabla^2_{2D} + \beta_1^2)\,A_z(r) = 0$ will be required to meet the boundary conditions at least when $\mu_1 \neq \mu_2$. The form of this homogeneous Helmholtz equation is

$(1/r)\partial_r(r\partial_r f) + \beta_1^2 f = 0$
$\partial_r(r\partial_r f) + r\beta_1^2 f = 0$
$rf'' + f' + r\beta_1^2 f = 0$
$f'' + (1/r)f' + \beta_1^2 f = 0$

which unsurprisingly is the same equation we encountered in (2.1.14) which has solutions of the type $J_0(\beta_1 r)$, $Y_0(\beta_1 r)$ with $\beta_1 = e^{j3\pi/4}\sqrt{\omega\mu_1\sigma_1}$. These are the Kelvin functions.

4. An alternative is to regard (M.5.1) as a 3D Helmholtz equation

$$(\nabla^2 + \beta_1^2)\, A_z(r,z) = -\,[I\mu_2/(\pi a^2)]\,\theta(r\le a) \qquad \mathbf{B} = B_\theta\hat{\boldsymbol{\theta}} \quad \text{with} \quad B_\theta = -\,\partial_r A_z\,. \qquad \text{(M.5.1)}$$

and then the 3D Helmholtz integral becomes (see Appendix H)

$$A_z^H(r,z) = \int dV'\,\left[\frac{1}{4\pi}\,\frac{e^{-j\beta R}}{R}\right]\,[I\mu_2/(\pi a^2)]\,\theta(r\le a)$$

$$= [I\mu_2/(\pi a^2)]\,\frac{1}{4\pi}\int_0^a r'\,dr'\int_{-\pi}^{\pi} d\theta'\int_{-\infty}^{\infty} dz'\,\frac{e^{-j\beta R}}{R} \qquad R^2 = r^2 + r'^2 - 2rr'\cos(\theta-\theta') + (z-z')^2$$

From (M.5.2) we expect the dz' integral here will be $4\pi\,(j/4)\,H_0^{(1)}(\beta_1 R)$ with $R^2 = r^2 + r'^2 - 2rr'\cos(\theta-\theta')$.