Appendix G (used to be M before 1.17.14) 11.24.13

Comments 12/5/13. This Appendix M was never installed. The opening section gives a good summary of what is done here. Part 1 still has my "illogical" problem which I have to consider valid only in the low-ω limit, but I suppose this is OK to do. In this doc, I am aware of the A_z boundary conditions and of the possible need for homo adder solutions. I was pretty excited about this Appendix M when I wrote it, but I feel cooler now and it is just on hold. I am using the King gauge global region R method, and it requires me to put in *something* for the driving current! This just makes it seem that the Helmholtz A_z equation is not a good way to solve the round wire problem for A_z in a precise manner. This King method only works if you have a prescribed current configuration.

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Appendix G: The DC vector potential of a round wire carrying a uniform current

In this problem, an isolated, infinitely-long and z-aligned round wire ($\mu_2,\varepsilon_2,\sigma_2$, radius a) carries a current I. The wire is immersed in an infinite dielectric medium ($\mu_1,\varepsilon_1,\sigma_1$). We begin for general ω, but quickly go to the DC limit ω = 0. We wish to calculate the vector potential **A** of this wire *both inside and outside*.

Section G.1 sets up the problem, makes some ansatz assumptions, and then ends up with a 2D Poisson equation for the potential which is $\nabla^2_{2D} A_z(r) = - [I\mu_2/(\pi a^2)]\ \theta(r \le a)$.

Section G.2 directly solves this Poisson equation for the potential $A_z(r)$. The solution is required to meet two boundary conditions at r = a.

Section G.3 very quickly computes this same $A_z(r)$ using Ampere's Law with the same boundary conditions and obtains the same result found in Section G.2.

Section G.4 laboriously obtains the same $A_z(r)$ result using the 2D Helmholtz integral (which in this case is really just a Poisson integral). This serves as a prototype case for dealing with such integrals, so much detail is provided. It is found that for $\mu_1 \neq \mu_2$ homogenous terms must be added to the Helmholtz integral in order to meet the boundary conditions.

Section G.5 comments on the solution for A_z at low frequency ω > 0.

G.1 Setup and Assumptions

The ω-domain Helmholtz wave equation for **A** using the King gauge is given by (1.5.4),

$$(\nabla^2 + \beta_1^2)\mathbf{A} = -\mu_2 \mathbf{J}_2 \qquad \beta_1^2 = \omega^2 \mu_1 \xi_1 \qquad \xi_1 \equiv \varepsilon_1 - j\sigma_1/\omega \qquad \text{(G.1.1)}$$

$$\text{div}\ \mathbf{A} = j\omega\mu_1\xi_1\varphi \ . \qquad \text{// King gauge} \qquad \text{(G.1.2)}$$

We take a uniform prescribed current inside the wire (assume low frequency),

$$\mathbf{J_2} = [I/(\pi a^2)]\ \hat{\mathbf{z}} \qquad \text{(G.1.3)}$$

so the Helmholtz wave equation reads

$$(\nabla^2 + \beta_1^2)\mathbf{A}(\mathbf{x}) = -\ [I\mu_2/(\pi a^2)]\ \theta(\sqrt{x^2+y^2} < a)\ \ \hat{\mathbf{z}}$$

where ∇^2 is the *vector* Laplacian and where $\theta(B) = 1$ if B is true, else 0.

Using Cartesian components, this says

$$(\nabla^2 + \beta_1^2)A_{\mathbf{x}}(\mathbf{x}) = 0$$
$$(\nabla^2 + \beta_1^2)A_{\mathbf{y}}(\mathbf{x}) = 0$$
$$(\nabla^2 + \beta_1^2)A_{\mathbf{z}}(\mathbf{x}) = -\ [I\mu_2/(\pi a^2)]\ \theta(\sqrt{x^2+y^2} < a)$$

where in *these* three equations ∇^2 is the *scalar* Laplacian. We shall seek a solution in which both $A_{\mathbf{x}}$ and $A_{\mathbf{y}}$ vanish. In this case we have

$$\mathbf{A}(\mathbf{x}) = A_{\mathbf{z}}(\mathbf{x})\ \hat{\mathbf{z}}\ .$$

We assume a very low frequency ω for which we know any longitudinal wave that might be going down the wire has a very long wavelength. We then ignore z variations in $A_{\mathbf{z}}$ to write

$$\mathbf{A}(\mathbf{x}) = A_{\mathbf{z}}(x,y)\ \hat{\mathbf{z}}\ . \qquad \text{(G.1.4)}$$

In this case, one finds that $\nabla^2 A_{\mathbf{z}} = \nabla^2_{2D}\ A_{\mathbf{z}}$ and then our only equation of interest is this:

$$(\nabla^2_{2D} + \beta_1^2)\ A_{\mathbf{z}}(x,y) = -\ [I\mu_2/(\pi a^2)]\ \theta(\sqrt{x^2+y^2} < a)\quad . \qquad \text{(G.1.5)}$$

We now take $\omega \to 0$ to get

$$\nabla^2_{2D}\ A_{\mathbf{z}}(x,y) = -\ [I\mu_2/(\pi a^2)]\ \theta(x^2+y^2 < a^2)$$

which is just a 2D Poisson equation with a constant source limited to a region of space. At this point we are free to replace x,y with polar coordinates r,θ, so we have for r in the range $(0,\infty)$,

$$\nabla^2_{2D}\ A_{\mathbf{z}}(r,\theta) =\ -\ [I\mu_2/(\pi a^2)]\ \theta(r<a)\ . \qquad \text{// } \theta(r<a) = \text{ Heaviside } \theta(a-r). \qquad \text{(G.1.6)}$$

From **B** = curl **A** in cylindrical coordinates we find that, since only $A_{\mathbf{z}}$ is non-vanishing,

$$\mathbf{B} = \text{curl } \mathbf{A} = \hat{\mathbf{r}}\ [\ r^{-1}\partial_{\theta}A_{\mathbf{z}} - \partial_{\mathbf{z}}A_{\theta}] + \hat{\boldsymbol{\theta}}\ [\partial_{\mathbf{z}}A_{\mathbf{r}} - \partial_{\mathbf{r}}A_{\mathbf{z}}] + \hat{\mathbf{z}}\ [\ r^{-1}\partial_{\mathbf{r}}(rA_{\theta}) - r^{-1}\partial_{\theta}A_{\mathbf{r}}\ ]$$

$= \hat{\mathbf{r}}\,[\, r^{-1}\partial_\theta A_z] + \hat{\boldsymbol{\theta}}\,[-\partial_r A_z]$.

Since we expect the magnetic field lines to be entirely in the $\hat{\boldsymbol{\theta}}$ direction, we are led to make the assumption that $\partial_\theta A_z = 0$ and then the problem is this

$$\nabla^2_{2D}\, A_z(r) = -\,[I\mu_2/(\pi a^2)]\;\theta(r\le a) \qquad \mathbf{B} = B_\theta\hat{\boldsymbol{\theta}} \quad \text{with} \quad B_\theta = -\,\partial_r A_z\,. \tag{G.1.7}$$

If we can find a solution, then our ansatz assumptions that $A_x = A_y = 0$ and $\mathbf{B} = B_\theta\hat{\boldsymbol{\theta}}$ are justified.

G.2 Direct solution for $A_z(r)$ from the differential equation

Using ∇^2_{2D} in polar coordinates our ODE (G.1.7) reads

$$(1/r)\partial_r(r\partial_r A_z(r)) = -\,[I\mu_2/(\pi a^2)]\;\theta(r\le a)$$

or

$$\partial_r(r\partial_r A_z(r)) = -\,[I\mu_2/(\pi a^2)]\; r\;\theta(r\le a)$$

or

$$r\,A_z''(r) + A_z'(r) = -\,[I\mu_2/(\pi a^2)]\; r\;\theta(r\le a)\,. \tag{G.2.1}$$

The two regional differential equations are then

$$\begin{aligned} &r\,A_z''(r) + A_z'(r) = 0 && r>a && \text{region 1}\\ &r\,A_z''(r) + A_z'(r) = -\,[I\mu_2/(\pi a^2)]\; r && r<a && \text{region 2}\,. \end{aligned} \tag{G.2.2}$$

The general-form solutions to these ODE's are,

$$\begin{aligned} &A_z(r) = C\,\ln(r) + D && r>a && \text{region 1}\\ &A_z(r) = -\,[I\mu_2/(4\pi a^2)]\; r^2 + E\,\ln(r) + F && r<a && \text{region 2} \end{aligned} \tag{G.2.3}$$

where there are 4 constants to be determined. For r>a the functions 1 and lnr are the well-known atomic forms (harmonic elements) for the 2D Laplace equation for situations of azimuthal symmetry. The first term in region 2 (G.2.3) is the particular solution of region 2 (G.2.2) to which we have added a possible homogeneous solution E ln(r) + F.

In order that $A_z(r)$ be finite at r = 0, we must set E = 0.

For very large r the round wire looks like a line source and we know the solution of *that* problem. Using Ampere's Law that $2\pi r H_\theta = I$ we find (recall that $B_\theta = -\,\partial_r A_z$)

$$H_\theta = [I/2\pi](1/r) \quad \Rightarrow \quad B_\theta = \mu_1\,[I/2\pi](1/r) \quad \Rightarrow \quad A_z = -\,[I\,\mu_1/2\pi]\,\ln(r) + D'\,. \qquad r \gg a$$

Comparing this solution to our r>a round wire solution we conclude that D' = D and $C = -[\mu_1 I/2\pi]$. There are still two unknown constants D and F :

$$A_z(r) = -[I\mu_1/2\pi]\,\ln(r) + D \qquad r > a \qquad \text{region 1}$$

$$A_z(r) = -[I\mu_2/(4\pi a^2)]\, r^2 + F \qquad r < a \qquad \text{region 2} \quad . \qquad \text{(G.2.4)}$$

The potential $A_z(r)$ always has an additive constant which we are free to specify and which affects nothing. We shall choose the zero point of $A_z(r)$ by setting D = 0 arbitrarily. This means that $A_z(r)$ has the simple form K ln(r) for r>a, and that choice implies that $A_z(a) = -[I\mu_1/2\pi]\ln(a)$, so we have in effect specified $A_z(r)$ on the wire surface to be this value. Notice for future reference that

$$\begin{aligned} \partial_r A_z(r) &= -[I\mu_1/2\pi]\,(1/r) && r > a && \text{region 1} \\ \partial_r A_z(r) &= -[I\mu_2/(2\pi a^2)]\, r && r < a && \text{region 2} \quad . \end{aligned} \qquad \text{(G.2.5)}$$

Now, since our prescribed current J_2 does *not* specify a free surface current K_z on the round wire surface, which would have the form

$$J_{s,surface} = K_z^{free}\ \delta(a-r),$$

we conclude that there is no free surface current on the round wire surface; there is only the bulk volume current $J_z = I/(\pi a^2)\theta(r<a)$. Therefore the boundary condition (1.1.46) applies (though now in polar coordinates) and we conclude that

$$(1/\mu_1)\,[(\partial_r A_z)(a)]^1 \;=\; (1/\mu_2)\,[(\partial_r A_z)(a)]^2 \ .$$

In addition, we shall require that A_z itself be continuous at the boundary, so here are our two boundary conditions of interest (superscript 1 means region 1 which is r>a),

$$\begin{aligned} &[A_z(a)]^1 = [A_z(a)]^2 \ . \\ &(1/\mu_1)\,[(\partial_r A_z)(a)]^1 \;=\; (1/\mu_2)\,[(\partial_r A_z)(a)]^2 \ . \end{aligned} \qquad \text{(G.2.6)}$$

We now require that both these boundary conditions be met by the A_z expressions of (G.2.4),

$$\begin{aligned} &[A_z(a)]^1 = [A_z(a)]^2 && \text{// (G.2.6) repeated} \\ &(1/\mu_1)\,[(\partial_r A_z)(a)]^1 \;=\; (1/\mu_2)\,[(\partial_r A_z)(a)]^2 \\ &\text{or} \\ &-[I\mu_1/2\pi]\ln(a) = -[I\mu_2/(4\pi a^2)]\, a^2 + F && \text{// insert expressions, set r = a} \\ &(1/\mu_1)\{-[I\mu_1/2\pi]\,(1/a)\} \;=\; (1/\mu_2)\{-[I\mu_2/(2\pi a^2)]a\} \\ &\text{or} \\ &-[I\mu_1/2\pi]\ln(a) = -[I\mu_2/(4\pi)] + F && \text{// simplify} \\ &1 = 1 \end{aligned}$$

The second boundary condition is thus met automatically by our solution. The first says

$$F = [I\mu_2/(4\pi)] - [I\mu_1/2\pi]\ln(a)$$

and so the solution is then

$A_z(r) = -[I\mu_1/2\pi]\ln(r)$ $\quad r > a \quad$ region 1

$A_z(r) = -[I\mu_2/(4\pi a^2)]\, r^2 + [I\mu_2/(4\pi)] - [I\mu_1/2\pi]\ln(a)$ $\quad r < a \quad$ region 2

or

$A_z(r) = -[I\mu_1/2\pi]\ln(r)$ $\quad r > a \quad$ region 1

$A_z(r) = -[I\mu_2/(4\pi a^2)]\,(r^2-a^2) - [I\mu_1/2\pi]\ln(a)$ $\quad r < a \quad$ region 2 . (G.2.7)

This then is the complete solution to the problem for the round wire,

$\nabla^2_{2D} A_z(r) = -[I\mu_2/(\pi a^2)]\,\theta(r \le a)$ $\qquad \mathbf{B} = B_\theta \hat{\boldsymbol{\theta}}$ with $B_\theta = -\partial_r A_z$. *(G.1.7)*

where the potential is "pinned" by the requirement that $A_z = K \ln(r)$ for $r > a$.

We may now compute the B field from our potential solution,

$B_\theta = -\partial_r A_z = [I\mu_1/2\pi]\partial_r \ln(r) = [I\mu_1/2\pi](1/r)$ $\quad r > a \quad$ region 1

$B_\theta = -\partial_r A_z = [I\mu_2/(2\pi a^2)]\, r = [I\mu_2/2\pi](r/a^2)$ $\quad r < a \quad$ region 2 (G.2.8)

Comment: Although only μ_2 appears in the differential equation (G.1.7), once the equation is properly solved with attention to boundary conditions, we find that μ_2 appears in B_θ in region 1, while μ_1 appears in B_θ in region 2 ! This is the main point of this Section G.2.

G.3 Instant solution for A using Ampere's Law and computation of J_m

For $r > a$ Ampere's law (1.1.37) (converted to ω space with $\omega = 0$) says $2\pi r H_\theta = I$ so

$H_\theta = I/(2\pi r) \quad \Rightarrow \quad B_\theta = \mu_1 I/(2\pi r)$

$r>a$ $\quad$ region 1 (G.3.1)

$\Rightarrow -\partial_r A_z = \mu_1 I/(2\pi r) \quad \Rightarrow \quad A_z = -[\mu_1 I/(2\pi)]\ln(r) + D$.

For $r < a$ Ampere's law says $2\pi r H_\theta = I(\pi r^2/\pi a^2)$ [the "current enclosed"]

$H_\theta = I(r^2/a^2)1/(2\pi r) = I\, r/(2\pi a^2) \quad \Rightarrow \quad B_\theta = [I\mu_2/2\pi](r/a^2)$

$r<a$ $\quad$ region 2 (G.3.2)

$\Rightarrow -\partial_r A_z = [I\mu_2/2\pi](r/a^2) \quad \Rightarrow \quad A_z = -[I\mu_2/4\pi](r^2/a^2) + E$

We then set $D = 0$ to get $A_z = K \ln(r)$ for $r > 0$, as done previously, and then we must match at $r = a$:

$-[\mu_1 I/(2\pi)]\ln(a) = -[I\mu_2/4\pi] + E \quad \Rightarrow \quad E = [I\mu_2/4\pi] - [\mu_1 I/(2\pi)]\ln(a)$

so our potential solution is then

$A_z(r) = -[\mu_1 I/(2\pi)]\ln(r)$ $\quad r>a$

$A_z(r) = -[I\mu_2/4\pi](r^2/a^2) + \{[I\mu_2/4\pi] - [\mu_1 I/(2\pi)]\ln(a)\}$ $\quad r<a$

or

$$A_z(r) = -[\mu_1 I/(2\pi)] \ln(r) \qquad r>a$$
$$A_z(r) = -[I\mu_2/4\pi a^2]r^2 + [I\mu_2/4\pi] - [\mu_1 I/(2\pi)] \ln(a) \qquad r<a$$

or

$$A_z(r) = -[\mu_1 I/(2\pi)] \ln(r) \qquad r>a$$
$$A_z(r) = -[I\mu_2/4\pi a^2](r^2-a^2) - [\mu_1 I/(2\pi)] \ln(a) \qquad r<a \qquad \text{(G.3.3)}$$

This result agrees with result (G.2.7) of the previous section.

Magnetization Current: It was mentioned in Section G.2 that there is no *free* surface current K_z^{free} at the wire surface. There is in fact a bound magnetization current on this surface and in both bulk regions as well. Luckily, our Helmholtz equation is unable to see these currents so we don't have to worry about them. The magnetization current density is given by $\mathbf{J_m}$ = curl **M** where **M** = $[\mu/\mu^0 - 1]$ **H** as shown in (1.1.20-23). Here then is the calculation:

$$M_\theta^1 = [\mu_1/\mu_0 - 1] H_\theta^1 = [\mu_1/\mu_0 - 1] (I/2\pi r) \qquad r > a$$
$$M_\theta^2 = [\mu_2/\mu_0 - 1] H_\theta^2 = [\mu_2/\mu_0 - 1] (Ir/2\pi a^2) \qquad r < a$$

or

$$M_\theta = [\mu_1/\mu_0 - 1] (I/2\pi r)\, \theta(r>a) + [\mu_2/\mu_0 - 1] (Ir/2\pi a^2)\theta(r<a)$$

$$\partial_r M_\theta = [\mu_1/\mu_0 - 1] (I/2\pi r)\, \delta(r-a) + [\mu_2/\mu_0 - 1] (Ir/2\pi a^2)[-\delta(r-a)] = [\mu_1/\mu_0 - \mu_2/\mu_0] (I/2\pi r)\, \delta(r-a)$$

$$\mathbf{J_m} = \text{curl } \mathbf{M} = \hat{\mathbf{z}} [\, r^{-1}\partial_r\{rM_\theta\}\,] \qquad \text{// see below (G.1.6) above}$$

$$J_{mz} = r^{-1}\partial_r\{rM_\theta\} = r^{-1}[M_\theta + r\partial_r M_\theta] = r^{-1}M_\theta + \partial_r M_\theta \qquad \text{(G.3.4)}$$

$$= [\mu_1/\mu_0 - 1] (I/2\pi r^2)\, \theta(r>a) + [\mu_2/\mu_0 - 1] (I/2\pi a^2)\theta(r<a) + [\mu_1 - \mu_2] (I/2\pi a\mu_0)\, \delta(r-a)$$
$$\sim 1/r^2 \text{ outside} \qquad \text{constant inside} \qquad \text{on the surface}$$

It is the slope discontinuity of the bulk magnetization at the surface which creates the surface current term in J_{mz}.

G.4 Solution using the 2D Helmholtz Integral

This method is technically more difficult than the first two methods, so we show a lot of detail. The method is important because our entire Chapter 4 is based on using Helmholtz integrals to compute transmission line parameters, and this is one Helmholtz integral that we can actually compute without *too* much effort. An important result we find is that, when $\mu_1 \neq \mu_2$, the Helmholtz integral by itself does not supply the complete solution, and one must add in some amount of homogeneous solution of $\nabla^2_{2D} A_z(r) = 0$ in order to meet the required boundary conditions at r = a. This subject is then pursued more in Section G.x. We really have a Poisson integral since $\beta_1^2 = 0$, but the full Helmholtz integral works the same way so we keep referring to it as a Helmholtz integral.

Recall from above:

$$\nabla^2_{2D} A_z(r) = - [I\mu_2/(\pi a^2)]\ \theta(r \le a) \qquad \mathbf{B} = B_\theta \hat{\boldsymbol{\theta}} \quad \text{with} \quad B_\theta = - \partial_r A_z \,. \qquad \textit{(G.1.7)}$$

Using the 2D free-space Green's function (propagator) as reviewed in Appendix I equation (I.1.6),

$$g(x|x') = (1/2\pi) \ln(1/R) = - (1/4\pi) \ln(R^2) \qquad R = R = |\mathbf{x}-\mathbf{x}'| \qquad (G.4.1)$$

we may write the particular solution to (G.1.7) as the following "Helmholtz" integral [see (I.1.8)]

$$A_z^H(r) = - \int dA' \, [(1/4\pi) \ln(R^2)] \, [I\mu_2/(\pi a^2)]\ \theta(r \le a)$$

$$= - \int_0^a r' \, dr' \int_{-\pi}^{\pi} d\theta' \, [(1/4\pi) \ln(r^2 + r'^2 - 2rr'\cos(\theta-\theta'))] \, [I\mu_2/(\pi a^2)]$$

$$= - (1/4\pi) \, [I\mu_2/(\pi a^2)] \int_0^a r' \, dr' \, [\int_{-\pi}^{\pi} d\theta' \ln(r^2 + r'^2 - 2rr'\cos(\theta-\theta')) \,]$$

$$= - (1/4\pi) \, [I\mu_2/(\pi a^2)] \int_0^a r' \, dr' \, [2 \, I_1(r,r') \,] \qquad (G.4.2)$$

where

$$I_1(r,r') \equiv (1/2) \int_{-\pi}^{\pi} d\theta' \ln(r^2 + r'^2 - 2rr'\cos(\theta-\theta')) = (1/2) \int_{-\pi}^{\pi} d\theta'' \ln(r^2 + r'^2 - 2rr'\cos(\theta''))$$

$$= \int_0^{\pi} dx \ln(r^2 + r'^2 - 2rr'\cos x) \,. \qquad (G.4.3)$$

Since this integral I_1 is quite important, we evaluate it two independent ways:

Method 1:

$$I_1 = \int_0^{\pi} dx \ln[r^2(1 + r'^2/r^2 - 2r'/r\cos x) \,]$$

$$= \int_0^{\pi} dx \ln[r^2(1 + \alpha^2 - 2\alpha\cos x) \,] \qquad \alpha \equiv r'/r$$

$$= \int_0^{\pi} dx \ln[r^2] + \int_0^{\pi} dx \ln(1 + \alpha^2 - 2\alpha \cos x)$$

$$= \pi \ln(r^2) + I_2 \,. \qquad (G.4.4)$$

We then use this easily derived expansion Stakgold Vol II p 104 (6.22),

$$\ln(1 + \alpha^2 - 2\alpha \cos x) = -2 \sum_{n=1}^{\infty} (\alpha^n/n)\cos(nx) \qquad |\alpha| < 1 \qquad \text{(G.4.5)}$$

which is just a Fourier Cosine Series expansion of $\ln(1 + \alpha^2 - 2\alpha \cos x)$. Then,

$$I_2 = \int_0^{\pi} dx \ln(1+ \alpha^2 - 2\alpha\cos x) = -2 \sum_{n=1}^{\infty}(\alpha^n/n) \int_0^{\pi} dx \cos(nx)$$

$$= -2 \sum_{n=1}^{\infty}(\alpha^n/n) [(1/n)\sin(nx)]|^{\pi}_0 = -2 \sum_{n=1}^{\infty}(\alpha^n/n^2) [\sin(n\pi) - \sin(0)] = 0$$

and then

$$I_1 = \pi \ln(r^2) + I_2 = \pi \ln(r^2) + 0 = 2\pi \ln(r) \qquad \alpha < 1 \quad \Leftrightarrow \quad r'<r$$

Since $I_1(r,r')$ is symmetric under $r \leftrightarrow r'$ we must have

$$I_1 = 2\pi \begin{cases} \ln(r) & r' < r \\ \ln(r') & r < r' \end{cases} = 2\pi \begin{cases} \ln(r) & r' < r \\ \ln(r') & r' > r \end{cases} . \qquad \text{(G.4.6)}$$

Method 2:

$$I_1 = \int_0^{\pi} dx \ln(r^2 + r'^2 - 2rr'\cos x) = \int_0^{\pi} dx \ln(a + b\cos x) .$$

This integral may be evaluated using GR7 p 531 4.224,

$$9.^8 \qquad \int_0^{\pi} \ln(a + b\cos x)\, dx = \pi \ln \frac{a + \sqrt{a^2 - b^2}}{2} \qquad [a \geq |b| > 0] \qquad \text{GW (322)(15)}$$

with $a = r'^2 + r^2$ and $b = -2rr'$ and $a^2 - b^2 = (r'^2 - r^2)^2$ so that $\sqrt{a^2 - b^2} = | r'^2 - r^2 |$. The condition $a > |b| > 0$ is met since $(r \pm r')^2 > 0 \Rightarrow r^2 + r'^2 > \pm 2rr'$ which says $a > \pm b$ so $a > |b|$. Thus,

$$I_1 = \int_0^{\pi} dx \ln [r'^2 + r^2 - 2rr' \cos x)] = \pi \ln \left[\frac{(r'^2 + r^2) + | r'^2 - r^2 |}{2} \right] = \begin{cases} \pi \ln [r'^2] & r' > r \\ \pi \ln [r^2] & r' < r \end{cases}$$

$$= 2\pi \begin{cases} \ln(r) & r' < r \\ \ln(r') & r' > r \end{cases} \qquad \text{(G.4.7)}$$

which agrees with the result of method 1.

Using the notation $\theta(x > a) \equiv \theta(x - a)$ [Heaviside step] we write (G.4.7) as

$$I_1 = 2\pi \{ \ln(r)\theta(r'<r) + \ln(r')\, \theta(r'>r) \} . \qquad \text{(G.4.8)}$$

Note: We use notation $\theta(x > a)$ in place of the more correct $\theta(x-a)$ because it requires "less thinking" to interpret.

Our particular "Helmholtz" integral (G.4.2) is then

$$A_z^H(r) = - (1/4\pi)\, I\mu_2/(\pi a^2) \int_0^a r'\, dr' \, [\, 2\, I_1(r,r') \,]$$

$$= - (1/4\pi)\, I\mu_2/(\pi a^2) \int_0^a r'\, dr' \; 4\pi \, \{ \ln(r)\theta(r'<r) + \ln(r')\, \theta(r'>r) \}$$

$$= - I\mu_2/(\pi a^2) \int_0^a r'\, dr' \, \{ \ln(r)\theta(r'<r) + \ln(r')\, \theta(r'>r) \}$$

$$= - I\mu_2/(\pi a^2) \, \{ \ln(r) \int_0^a r'\, dr'\, \theta(r'<r) + \int_0^a r'\, dr'\, \ln(r')\, \theta(r'>r) \} \quad . \qquad \text{(G.4.9)}$$

The two integrals are

$$\int_0^a r'\, dr'\, \theta(r'<r) = \int_0^{\min(a,r)} r'\, dr' = (1/2)\, [\min(a,r)]^2 \qquad \text{(G.4.10)}$$

$$\int_0^a r'\, dr'\, \ln(r')\, \theta(r'>r) = \theta(r<a) \int_r^a r'\, dr'\, \ln(r') = \theta(r<a) \int_r^a x \ln x \, dx$$

$$= \theta(r<a)\, [(1/4)\, (r^2-a^2) + (1/2)\, a^2 \ln a - (1/2)\, r^2 \ln r \,] \quad . \qquad \text{(G.4.11)}$$

Then

$$A_z^H(r) = - I\mu_2/(\pi a^2) \, \{ \ln(r) \int_0^a r'\, dr'\, \theta(r'<r) + \int_0^a r'\, dr'\, \ln(r')\, \theta(r'>r) \}$$

$$= - I\mu_2/(\pi a^2) \, \{ \ln(r)\, (1/2)\, [\min(a,r)]^2 + \theta(r<a)\, [(1/4)\, (r^2-a^2) + (1/2)\, a^2 \ln a - (1/2)\, r^2 \ln r \,] \} \, .$$

For $r > a$ we get

$$A_z^H(r) = - I\mu_2/(\pi a^2) \, \{ \ln(r)\, (1/2) a^2 \} = - I\mu_2/(2\pi) \ln(r)$$

$$\partial_r A_z^H(r) = - I\mu_2/(2\pi)\, (1/r) \, . \qquad \text{(G.4.12)}$$

For $r < a$ the result is

$$A_z^H(r) = - I\mu_2/(\pi a^2) \, \{ \ln(r)\, (1/2) r^2 + (1/4)\, (r^2-a^2) + (1/2)\, a^2 \ln a - (1/2)\, r^2 \ln r \}$$

$$= - I\mu_2/(\pi a^2) \, \{ (1/4)\, (r^2-a^2) + (1/2)\, a^2 \ln a \} \qquad \text{(G.4.13)}$$

$\partial_r A_z^H(r) = - I\mu_2/(\pi a^2)\,(r/2)$.

Recall the two boundary conditions

$$[A_z(a)]^1 = [A_z(a)]^2 .$$
$$(1/\mu_1)\,[(\partial_r A_z)(a)]^1 = (1/\mu_2)\,[(\partial_r A_z)(a)]^2 . \qquad (G.2.6)$$

For our particular Helmholtz integral $A_z^H(r)$ we evaluate these boundary conditions to find

$$- I\mu_2/(2\pi)\,\ln(a) = - I\mu_2/(\pi a^2)\,(1/2)\,a^2 \ln a$$
$$(1/\mu_1)\,[- I\mu_2/(2\pi)\,(1/a)] = (1/\mu_2)\,[- I\mu_2/(\pi a^2)\,(a/2)]$$

or

$$1 = 1$$
$$(\mu_2/\mu_1) = 1$$

Thus, only in the case $\mu_1 = \mu_2$ does the Helmholtz particular solution meet both boundary conditions. If $\mu_1 \neq \mu_2$, we must add to the particular solution some amount of homogeneous solution of $\nabla^2_{2D} A_z(r,\theta) = 0$. So we then write generally,

$$A_z(r) = A_z^H(r) + A_z^{homo}(r) \qquad (G.4.14)$$

where we know that $A_z^{homo}(r)$ can only have terms $\alpha + \beta \ln r$. We then write for the two regions

$$A_z(r) = - I\mu_2/(2\pi)\,\ln(r) + \alpha + \beta \ln r \qquad r>a$$
$$A_z(r) = - I\mu_2/(\pi a^2)\,\{\,(1/4)\,(r^2-a^2) + (1/2)\,a^2 \ln a\,\} + \alpha' + \beta' \ln r \qquad r<a$$

As earlier, we choose the zero point for $A_z(r)$ by requiring that the large r behavior be K ln(r) without a constant added, which then means $\alpha = 0$. And for r<a we must have $\beta' = 0$ to be finite at r = 0. So

$$A_z(r) = [- I\mu_2/(2\pi) + \beta] \ln r \qquad r>a$$
$$A_z(r) = - I\mu_2/(\pi a^2)\,\{\,(1/4)\,(r^2-a^2) + (1/2)\,a^2 \ln a\,\} + \alpha' \qquad r<a$$

(G.4.15)

$$-\partial_r A_z(r) = [I\mu_2/(2\pi) - \beta](1/r) \qquad r>a$$
$$-\partial_r A_z(r) = I\mu_2/(\pi a^2)\,(2r) \qquad r<a$$

The boundary conditions are then

$$[A_z(a)]^1 = [A_z(a)]^2$$
$$(1/\mu_1)\,[(-\partial_r A_z)(a)]^1 = (1/\mu_2)\,[(-\partial_r A_z)(a)]^2 \qquad \text{// (G.2.6) repeated}$$

or

$$[- I\mu_2/(2\pi) + \beta] \ln a = - I\mu_2/(\pi a^2)\,\{\,(1/4)\,(a^2-a^2) + (1/2)\,a^2 \ln a\,\} + \alpha'$$
$$(1/\mu_1)[\,I\mu_2/(2\pi) - \beta](1/a) = (1/\mu_2)\,I\mu_2/(\pi a^2)\,(2a) \qquad \text{// insert expressions, set r = a}$$

or

$$[- I\mu_2/(2\pi) + \beta] \ln a = - I\mu_2/(2\pi) \ln a + \alpha'$$

$(1/\mu_1)[\ I\mu_2/(2\pi) - \beta](1/a) = I/(2\pi a)$ // simplify

or

$\beta \ln a = \alpha'$

$I\mu_2/(2\pi) - \beta = \mu_1 I/(2\pi)$ // simplify some more

so we find that

$$\beta = I/(2\pi)\ (\mu_2-\mu_1)$$
$$\alpha' = I/(2\pi)\ (\mu_2-\mu_1)\ \ln a \qquad \text{(G.4.16)}$$

The full solution is then

$$A_z(r) = [- I\mu_2/(2\pi) + \beta]\ \ln r \qquad r>a$$
$$A_z(r) = - I\mu_2/(\pi a^2)\ \{ (1/4)\ (r^2-a^2) + (1/2)\ a^2 \ln a \}\ + \alpha' \qquad r<a$$

or

$$A_z(r) = [- I\mu_2/(2\pi) + \{I/(2\pi)\ (\mu_2-\mu_1)\}]\ \ln r \qquad r>a$$
$$A_z(r) = - I\mu_2/(\pi a^2)\ \{ (1/4)\ (r^2-a^2) + (1/2)\ a^2 \ln a \}\ + \{ I/(2\pi)\ (\mu_2-\mu_1)\ \ln a \} \qquad r<a$$

or

$$A_z(r) = I/(2\pi)\ [- \mu_2 + (\mu_2-\mu_1)]\ \ln r \qquad r>a$$
$$A_z(r) = - I\mu_2/(\pi a^2)\ (1/4)\ (r^2-a^2) - I\mu_2/(\pi a^2)\ (1/2)\ a^2 \ln a + I/(2\pi)\ (\mu_2-\mu_1)\ \ln a \qquad r<a$$

or

$$A_z(r) = - (I/(2\pi)\ [\mu_1]\ \ln r \qquad r>a$$
$$A_z(r) = - I\mu_2/(\pi a^2)\ (1/4)\ (r^2-a^2) - I\mu_1/(\pi)\ (1/2)\ \ln a \qquad r<a$$

or

$$A_z(r) = - [I\mu_1/2\pi]\ \ln r \qquad r>a$$
$$A_z(r) = - [I\mu_2/(4\pi a^2)]\ (r^2-a^2) - [I\mu_1/2\pi]\ \ln a \qquad r<a \qquad \text{(G.4.17)}$$

This result matches the results (G.2.7) and (G.3.3) of the previous two methods and then results in the B field solution

$$B_\theta = - \partial_r A_z = [I\mu_1/2\pi]\partial_r \ln(r) = [I\mu_1/2\pi](1/r) \qquad r > a \quad \text{region 1}$$
$$B_\theta = - \partial_r A_z = [I\mu_2/(2\pi a^2)]\ r = [I\mu_2/2\pi](r/a^2) \qquad r < a \quad \text{region 2} \qquad \textit{(G.2.8)}$$

G.5 Comments on the low frequency solution

In Chapter 2 we compute the E and B fields inside a round wire operating at frequency ω. The results are rather complicated and involve special Bessel functions called Kelvin functions. The vector potential was not used in that Chapter. We shall *not* carry through the calculation here, but shall merely make some comments.

1. For sufficiently low frequencies (the transmission line limit) we imagine that the ansatz assumptions we made in Section G.1 are still pretty good. The current distribution will be nearly uniform. There will likely be some small A_x and A_y fields which we can ignore, and we still assume roughly that $\mathbf{B} = B_\theta \hat{\boldsymbol{\theta}}$ with $B_\theta = - \partial_r A_z$ and that we can ignore the z-dependence of A_z, though we know it must vary some

small amount in order to have a long-λ wave passing down the wire. Therefore, our problem is basically (G.1.5) for small β_1^2,

$$(\nabla^2_{2D} + \beta_1^2)\, A_z(r) = -\,[I\mu_2/(\pi a^2)]\;\theta(r\le a) \qquad \mathbf{B} = B_\theta\hat{\boldsymbol{\theta}} \;\text{ with }\; B_\theta = -\,\partial_r A_z\,. \qquad \text{(G.5.1)}$$

2. Since $\nabla^2_{2D} = (1/r)\partial_r(r\partial_r)$, one could write out the above differential equation and repeat the work of section G.2 above. The resulting B field obtained from $B_\theta = -\,\partial_r A_z$ should then agree with the low frequency limit of (2.2.25) which applies inside the round wire,

$$B_\theta(r) = B_\theta(a)\,\frac{J_1(\beta_1 r)}{J_1(\beta_1 a)}\quad . \qquad \textit{(2.2.25)}$$

That low-frequency limit is

$$B_\theta(r) \approx B_\theta(a)\,\frac{\beta_1 r/2 - (\beta_1 r)^3/16}{\beta_1 a/2 - (\beta_1 a)^3/16} = B_\theta(a)\,\frac{r - \beta_1^2 r^3/8}{a - \beta_1^2 a^3/8} \qquad \beta_1^2 = \omega^2\mu\;\xi_1 \qquad \text{(G.5.2)}$$

Certainly as $\omega \to 0$ so $\beta_1 \to 0$ the result $B_\theta(r) = B_\theta(a)(r/a)$ agrees with (G.2.8).

3. The 2D free-space Helmholtz Green's function propagator is shown in (I.1.7) to be

$$g(x|x') = (j/4)\,H_0^{(1)}(\beta_1 R) \qquad R = R = |\mathbf{x}-\mathbf{x}'|$$

where $H_0^{(1)}$ is a Hankel function. Thus, we may write the particular solution to (G.5.1) as the following Helmholtz integral [see (I.1.9)] ,

$$A_z^H(r) = \int dA'\,[(j/4)\,H_0^{(1)}(kR)\,]\;[I\mu_2/(\pi a^2)]\;\theta(r\le a) \qquad R = R = |\mathbf{x}-\mathbf{x}'|$$

$$= \;[I\mu_2/(\pi a^2)]\,(j/4)\int_0^a r'\,dr'\int_{-\pi}^{\pi} d\theta'\,H_0^{(1)}\!\left(\beta_1\sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')}\,\right)\;. \qquad \text{(G.5.3)}$$

The dθ' integral is actually doable with this result (making use of GR7 p 726 6.684 1 and 2)

$$\int_{-\pi}^{\pi} d\theta'\,H_0^{(1)}\!\left(\beta_1\sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')}\,\right)$$

$$= (1/2)\,\{\,\pi\,J_0(\beta_1 r)\,H_0^{(1)}(\beta_1 r')\theta(r'>r) + \pi\,J_0(\beta_1 r')\,H_0^{(1)}(\beta_1 r)\theta(r'<r)\,\}\;. \qquad \text{(G.5.4)}$$

The two dr' integrals can then be done (using GR7 p 629-630 Section 5.5) with the final result

$$A_z^H(r) = [I\mu_2/(\pi a^2)](j/4)2\pi\;(1/\beta_1)\,*$$
$$\{\,J_0(\beta_1 r)\,\theta(a>r)\,[a\,H_1^{(1)}(\beta_1 a) - r\,H_1^{(1)}(\beta_1 r)\,] + H_0^{(1)}(\beta_1 r)\begin{cases} r\,J_1(\beta_1 r) & r<a \\ a\,J_1(\beta_1 a) & r>a \end{cases}\,\}\;. \qquad \text{(G.5.5)}$$

We leave it to the reader to determine the small β_1 limit of this result and see if the resulting $B_\theta = -\partial_r A_z$ agrees with (G.5.2) after homogeneous solutions are added to match boundary conditions. Remember that we only expect this result to be meaningful for low ω since we have assumed the uniform current distribution of (G.1.3).

If one makes the small-argument approximation $H_0^{(1)}(x) \approx (2j/\pi)\ln(x)$ directly in (G.5.3), the integral replicates the Poisson result (G.4.2), so more expansion terms would be needed for this approach.