Dealing with Homo Solutions in a Transmission Line PhL 11.24.13

Comments 12/5/13. This doc shows how I was struggling with the issues related to the homo solutions question and why King seems to do all his work never adding homo solutions and just using the Helmholtz integral for A_z. I first wondered if it was the fact that total current is 0 in a full transmission line and maybe that makes the homo terms go away (no, that is not it). I now think it is the fact that in his analysis, there is only one μ for dielectric and conductors, and in this case, the A_z and $\partial_n A_z$ boundary conditions are "automatically met" by the Helmholtz integral, so there is no need to add homo solutions. This is not the case when $\mu_c \neq \mu_d$, and this is a case King never treats.

I want to be able to ignore homo solutions somehow so I can use the Helmholtz integral by itself to compute parameters of a transmission line.

Background: In my off-line analysis of a round wire with a uniform current (which is the low frequency limit of an actual wire with AC), I found that the particular integral does NOT give the entire solution. You have to add a homo solution A ln(r) outside, and a homo solution B inside, and then match boundary conditions to determine A and B.

Plan A. Is it somehow the sum of the two conductors that makes the homo terms go away?

Now here is one argument that might be a hint. Consider a transmission line with two parallel conductors not one inside the other. They are not round. Suppose in the dielectric region there is some homo solution required to match the BC's. Write that as (for φ or A_z)

$$\varphi_{homo}(\mathbf{x}) = A\, F(\mathbf{x})$$

where F(x) is some complicated function perhaps for a weird shaped wire cross section. As we move very far away laterally from the transmission line, the two conductors "look like" round wires going down the central axis of a cyl coord system. In this large r case, we know that, if the currents are balanced, there is no ln(r) term in the total solution. So we know that

limit(large r) of $[\, \varphi_{part} + \varphi_{homo} \,] \neq$ proportional to ln(r) in dielectric [ok]

Now in this limit, I know that for a single round wire with uniform current, both φ_{part} AND φ_{homo} have ln(r) terms in them. I expect that to be true more generally for non-round wires. So I am claiming that when we include all conductors and include part + homo, there will be no ln(r) term. This does not say $\varphi_{homo} = 0$. So you could have, for large r, some φ_{homo} which cancels out φ_{part} in terms of the ln(r) component.

What I do know is this: in the distance region, if there IS a φ_{homo} term present, it can only have the general form A ln(r) + B in the large r limit. This is because in that limit, these are the only atoms for a solution of the 2D Laplace equation. Somehow I think I can argue that both A and B must be zero for a current balanced line. Let's assume that for the moment. Could probably show it for twin lead. So I guess my claim is this

(1) $\varphi_{homo} \rightarrow A \ln r + B$ for large r only possibility

(2) $A = 0$ and $B = 0$

(3) $\varphi_{homo} = C\,F(x)$ for non-large r, function F(x) complicated

Then I guess I am saying

limit large r [C F(x)] = 0

Either C = 0 or $F(x) \rightarrow 0$. Write an expansion for F(x) in some arbitrary cyl system whose center line is near the two conductors. In this expansion, we have an infinite series of terms with atoms ln r, 1, r, r^2, r^{-1}, r^{-2} and so on. I suppose we could have this general expansion.

limit large r [F(x)] = A lnr + B + $Cr^{-1} + Dr^{-2}$ ....

BUT, for large r, we have *axial* sym, which rules out things like C and D? This is a crucial question so be careful. I can see if two ways

(1) for large r, the terms $Cr^{-1} + Dr^{-2} \rightarrow 0$ even if C and D are not zero. At any finite r, there is some small amount of axial asym, so these terms could be present. So you cannot argue that C and D are 0.

(2) you can argue that C and D are 0.

Coaxial line. *Imagine* that dielectric exists both inside and outside. Then at large r, the above argument still holds and we conclude that $\varphi_{homo} = 0$.

So my conclusion is this: For a total current = 0 transmission line of any kind, $\varphi_{homo} = 0$. Now what about φ_{part} ? Since it is not a Laplace solution, the above argument does not work and it could be non-zero. Well let's play the hand. Consider φ_{part}. It is NOT a solution of Laplace, so I don't know that it has the Laplace

Discovery: On page 11 King says "the solutions (23) and (24) take account of the boundary conditions automatically". I read the following 2.5 pages and I see no attempt on his part of back up this claim! But I do hope it is true and that is really what I am trying to show here. That is, I am trying to show that you don't need to add any homo terms. At least King is telling me that what I want to be true is in fact true.

[somewhere later I show that this King claim is valid *if* $\mu_1 = \mu_2$ since then "nothing really happens" at the boundary in terms of magnetic things like B and A_z]

I think I am one right track above but I need a way to decide that (2) is the right answer.

Analogy with a Line Charge in Electrostatics

Digression. Consider a line source of charge in 2D. Gauss's Law says $2\pi r E_r = \lambda$ and so $E_r = \lambda/2\pi r$ and then $\varphi = -(\lambda/2\pi) \ln(r)$. Far away from the line charge, we have this Laplace solution and this is an allowed atomic form, and there is no constant (the other allowed form).

Now suppose the line charge is really a thin wire of some weird cross section. We know that outside the wire we can for sure write the potential using this kind of expansion

$$\varphi = A \ln r + B + \Sigma_{m=1}^{\infty} (A_m r^m + B_m r^{-m}) e^{im\varphi}$$

since this is the most general Laplace solution. We know at once that $A_m = 0$ since if A_m were not zero, we would have a power blow up of the potential. We know that in the limit of large r, we have $\varphi = A \ln r$. Does this somehow tell us that B and B_m vanish??? So this is perhaps a simpler problem to think about. I would say that for sure at least some B_m must NOT vanish, otherwise close in you would have an azisym solution which we know is not true for a square cross section wire! So we then have for our wire of non-round cross section

$$\varphi(r) = A \ln r + B + \Sigma_{m=1}^{\infty} B_m \; r^{-m} e^{im\varphi}$$

So for a single conductor, I think this is the potential outside in a 2D analysis with $B_m \neq 0$.

If we consider then two conductors closely spaced but opposite current, and cyl axis is some average center line say. Then we would have

$$\varphi(r) = A \ln r + B + \Sigma_{m=1}^{\infty} B_m \; r^{-m} e^{im\varphi}$$

$$\varphi'(r) = -A' \ln r - B' - \Sigma_{m=1}^{\infty} B'_m \; r^{-m} e^{im\varphi}$$

where the prime and nonprime coefficients are close in value. Then we have

$$\varphi_{tot} = \varphi + \varphi' = (A-A') \ln r + (B-B') + \Sigma_{m=1}^{\infty} (B_m - B_m') \; r^{-m} e^{im\varphi}$$

I just don't see how you can rule out these $(B_m - B_m')$ contributions! In fact, if we did an exact solution to a twin lead problem, I would guess we have some resulting coefficients.

Example: two point charges closely spaced, opposite charge. Then $\varphi \sim 1/r^2$ I think and $E \sim 1/r^3$ and this is an electric dipole field. There is certainly SOME power that survives far away.

So, any argument based on $B_m = 0$ is just plain wrong!

Status: I was hoping somehow the 2-conductor sum would magically make homo terms go away, but I was unable to show that.

Plan B. Still pondering the need to add homo terms.

I am now back to square 0. How does King know that BC's are automatically met? When I solve the single uniform round wire, I find that they are NOT automatically met and I have to add in some homo terms to make things work.

Reminder: Remember that King assumes in his (24) that all conductors have the same μ and that this is the same as the μ in the dielectric! At least with my derivation of things, that is what seems to be the case. You would wonder then why he talks so much about hysteresis and such.

Is it possible that King's μ is a variable depending on region? If that were the case, then the whole development would be different. I will run that right here. Here is (1.3.3) the starting point:

$$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2)\,\mathbf{A} = \text{grad}\,[\mu_1\varepsilon_1\,\partial_t\varphi + \text{div}\mathbf{A}\,] - \mu_1\mathbf{J} \quad . \qquad \text{// region 1} \qquad \textit{(1.3.3)}$$

In this region, I use the King gauge and end up with

$$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2 - \mu_1\sigma_1\partial_t)\,\mathbf{A} = 0 \qquad \text{// region 1} \qquad (1.3.19)$$

Fine. Now start over in region 2:

$$(\nabla^2 - \mu_2\varepsilon_2\,\partial_t^2)\,\mathbf{A} = \text{grad}\,[\mu_2\varepsilon_2\,\partial_t\varphi + \text{div}\mathbf{A}\,] - \mu_2\mathbf{J}_2 \qquad \text{// region 2} \qquad \textit{(1.3.3)}$$

Suppose I assume the King gauge in region 2 with region 2 parameters, [a rejected model]

$$\text{div}\,\mathbf{A} = -\,\mu_2\varepsilon_2\,\partial_t\varphi - \mu_2\sigma_2\varphi \qquad \text{// region 2}$$

Then here is what happens:

$$(\nabla^2 - \mu_2\varepsilon_2\,\partial_t^2)\,\mathbf{A} = \text{grad}\,[\mu_2\varepsilon_2\,\partial_t\varphi + \text{div}\mathbf{A}\,] - \mu_2\mathbf{J}_2 \qquad \text{// region 2} \qquad \textit{(1.3.3)}$$

$$(\nabla^2 - \mu_2\varepsilon_2\,\partial_t^2)\,\mathbf{A} = \text{grad}\,[\mu_2\varepsilon_2\,\partial_t\varphi + \{-\,\mu_2\varepsilon_2\,\partial_t\varphi - \mu_2\sigma_2\varphi\}\,] - \mu_2\mathbf{J}_2$$

$$(\nabla^2 - \mu_2\varepsilon_2\,\partial_t^2)\,\mathbf{A} = \text{grad}\,[-\,\mu_2\sigma_2\varphi\}\,] - \mu_2\mathbf{J}_2$$

$$(\nabla^2 - \mu_2\varepsilon_2\,\partial_t^2)\,\mathbf{A} = -\,\mu_2\sigma_2\;\text{grad}\,\varphi\,] - \mu_2\mathbf{J}_2$$

$$(\nabla^2 - \mu_2\varepsilon_2\,\partial_t^2)\,\mathbf{A} = -\,\mu_2\sigma_2\;\{-\mathbf{E} - \partial_t\mathbf{A}\}\,] - \mu_2\mathbf{J}_2$$

Since inside a conductor in region 2, set E = 0 as per usual arguments, then have

$$(\nabla^2 - \mu_2\varepsilon_2\,\partial_t^2)\,\mathbf{A} = -\,\mu_2\sigma_2\;\{-\,\partial_t\mathbf{A}\}\,] - \mu_2\mathbf{J}_2$$

$$(\nabla^2 - \mu_2\varepsilon_2\,\partial_t^2 - \mu_2\sigma_2\partial_t)\,\mathbf{A} = -\,\mu_2\mathbf{J}_2$$

But I thought that you got that damping term only if you "absorb" the conduction current. Well, that was necessary in region 1 because i could not there say that E = 0. There I ended up like this for RHS.

$$= -\,\mu_1\sigma_1\,(-\mathbf{E} - \partial_t\mathbf{A}) - \mu_1(\sigma_1\mathbf{E}) \;. \qquad \text{// from (1.3.1) and } \mathbf{J} = \sigma_1\mathbf{E}$$

$= - \mu_1\sigma_1 (-\partial_t \mathbf{A})$.

But in a conductor, maybe we can just set E = 0 as I have done above. Then we end up with this

$$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1\partial_t)\,\mathbf{A} = 0 \qquad \text{// region 1} \qquad (1.3.19)$$
$$(\nabla^2 - \mu_2\varepsilon_2 \partial_t^2 - \mu_2\sigma_2\partial_t)\,\mathbf{A} = - \mu_2\mathbf{J}_2 \qquad \text{// region 2}$$
$$(\nabla^2 - \mu_3\varepsilon_3 \partial_t^2 - \mu_3\sigma_3\partial_t)\,\mathbf{A} = - \mu_3\mathbf{J}_3 \qquad \text{// region 3}$$

The Helmholtz are then

$$(\nabla^2 + \beta_1^2)\,\mathbf{A} = 0$$
$$(\nabla^2 + \beta_2^2)\,\mathbf{A} = - \mu_2\mathbf{J}_2$$
$$(\nabla^2 + \beta_3^2)\,\mathbf{A} = - \mu_3\mathbf{J}_3$$

I think all is OK to this point. But now I cannot make a unified region R since the three wave operators are different. [correct] In region 1 which is the main interest, there is no source, so there is no Helmholtz integral! There is then no way in the world I can arrive at his (24) using this interpretation of the gauge. The only way I know is to "do it my way" and then it works.

Here is my result:

$$(\nabla^2 + \beta^2)\mathbf{A} = - \Sigma_{i=2}^{N}\, \mu_i\mathbf{J}_i \qquad \beta^2 = \omega^2\mu\,\xi$$

Now if you make all the conductors the same μ_c you would have

$$(\nabla^2 + \beta^2)\mathbf{A} = - \mu_c\, \Sigma_{i=2}^{N}\, \mathbf{J}_i \qquad \beta^2 = \omega^2\mu\,\xi$$

and then

$$\mathbf{A}(\mathbf{x},\omega) = \frac{\mu_c}{4\pi} \int [\,\Sigma_i\, \mathbf{J}_i(\mathbf{x}',\omega)]\, \frac{e^{-j\beta R}}{R}\, dV'$$

So maybe that is what King means. He means dielectric μ in β, and conductor μ out front. This μ_c is an overall factor, should not be too critical.

Now jump to my round wire analysis in section "Reminder of the Just-Fine...". Here is a summary

$$\nabla^2_{2D}\, A_z = 0 \qquad \text{region 1}$$
$$\nabla^2_{2D}\, A_z = - \mu_2\, J_{cz} \qquad \text{region 2}$$

$$A_z^2 = (1/2\pi) \int dV' \,[\mu_2 J_{cz}(\mathbf{x}')] \ln(1/R) = W = \text{Helmholtz integral}$$

$$W = -(I/2\pi)\,\mu_2 \begin{cases} \ln(r) & r > a \\ \ln a + (r^2/a^2 - 1)/2 & r < a \end{cases} \qquad \text{// by direct computation in App B I guess}$$

Notice that this DOES automatically satisfy the boundary condition $A_z^1(a) = A_z^2(a)$ without the addition of homo terms. And in fact $A_z^1(a) = -(I/2\pi)\,\mu_2 \ln(a)$ at the boundary, from either side. But it does NOT satisfy the 2nd boundary condition on $\partial_r A^z$. If I want to satisfy BOTH these boundary conditions assuming no free surface current on the conductor, I must first add these homo solutions

$A^1_{homo} = A \ln r$ // outside wire
$A^2_{homo} = d$ // inside wire

And then I match BC's and find that

$A = -\mu_1 (I/2\pi)$
$d = (I/2\pi)(\mu_2 - \mu_1)\ln a$

In particular, outside the wire I had to add this homo solution

$$A^1_{homo} = -\mu_1 (I/2\pi) \ln r$$

Even if $\mu_1 = \mu_2$, I still have to add the above homo solution in the dielectric region 1.

[I guess this is where I realized there is a slope condition on A_z at a boundary]

So where does my second BC come from? I will derive it right here in cyl coordinates.

$$\text{curl } \mathbf{A} = \hat{\mathbf{r}}\,[\, r^{-1}\partial_\theta A_z - \partial_z A_\theta] + \hat{\boldsymbol{\theta}}\,[\partial_z A_r - \partial_r A_z] + \hat{\mathbf{z}}\,[\, r^{-1}\partial_r(rA_\theta) - r^{-1}\partial_\theta A_r\,]$$

Assume that $\mathbf{A} = A_z(r,z)\,\hat{\mathbf{z}}$ does not depend on θ. Then

$$\text{curl } \mathbf{A} = \hat{\mathbf{r}}\,[\, r^{-1}\partial_\theta A_z] + \hat{\boldsymbol{\theta}}\,[-\partial_r A_z] = \hat{\boldsymbol{\theta}}\,[-\partial_r A_z] \qquad \text{so } B_\theta = -\partial_r A_z$$

Now from (1.1.44)

$$(1/\mu_2)B_{\theta 2} - (1/\mu_1)B_{\theta 1} = K_z^{free}$$

and then

$$(1/\mu_2)[-\partial_r A_z]^2 - (1/\mu_1)[-\partial_r A_z]^1 = K_z^{free}$$

$$(1/\mu_2)[\partial_r A_z]^2 - (1/\mu_1)[\partial_r A_z]^1 = -K_z^{free} \qquad \text{[here it is !]}$$

Thus I either have this BC or I have to have a free surface current. But that free surface current is not part of my prescribed current in the wire! So there is no free current by assumption, so this BC has to be met.

Then I have to add my homo term in the dielectric. Maybe I should write this up fully, but I have done it a lot of times.

You know, King's comments about BC's really only applies to the sum over all conductors, but you should be able to apply his (23) and (24) to a single conductor. He says "arbitrary configurations of conductors". So I think he claims boundary conditions are "automatically met" for a single conductor.

He does restrict his (23) and (24) to the calculation of φ and A outside the conductors. So if you compute these solutions, you will get some value for A_z as you approach any point on any conductor, so in that sense it meets this boundary condition. I don't like it! I don't believe his claim. I have a clean counterexample. So maybe he just means by "boundary conditions" the fact that A_z approaches some value on each of the plural conductors. That is true, but I argue that it is not the right value!

So how then does this "work out" in his first example? He basically writes (23) and (24) as (10a) and (10b) for V and W where these are differences, and he has assumed no homo contribution here. Since I am in the DC limit I will grant that only his k_0 survives. In his (16) the first integral is well defined as in (19), but there is also a second integral shown. He works hard to argue that this integral goes away in the DC limit with order β^2.

So to summarize: if you take two round wires, and if you ignore homo solutions, and if you do the full 3D analysis which he does, and you deal with the differences V and W, then you get "the right answer" and no homo terms were needed.

Now suppose we were to add the homo terms anyway which I know are there for each wire.

Well, maybe I need to solve this problem using my 2D propagator stuff and see what happens in that case. That would be the n=2 problem for this issue! BUT, in the line problem, perhaps you don't have uniform current in each conductor, so my approach would be wrong? But even he assumes b >> a and that currents are uniform.

Now what has happened to his transverse integral in (10)? That is all jammed into his I current.

So maybe I need to look at a fancier King example. He does that fat wires on page 24, but his approach here is to write my homo wave equation in region 1 which is $(\nabla^2 + \beta^2)\mathbf{A_z} = 0$. He then separates variables and ends up with 2D Helm equation which in my DC limit becomes a 2D Laplace equation and then goes and solves THAT for the fat wires. Nowhere to be seen are his (23) and (24) integrals!!!

Let's look now at his "3 phase" transmission line calculation on page 40. Here the three currents treated as complex in ω plane do in fact add up to 0 current if you add those three vectors at 120 degrees. And here he *does* use the Helmholtz integral for A_z.

I just searched the book for "homogeneous" and could find no reference to my issue. Mostly the word refers to the nature of a medium.

OK, I see my puzzlement. King really does assume that μ is the same for dielectric and for conductors and that is the only way that his A_z (23) or (24) work. In this case, his A_z solution *does* "automatically"

meet BC's at a boundary, because if $\mu_1 = \mu_2$ at the boundary, then there really IS no boundary and no reason for A_z to have a slope break there. I think this is a strong limitation of his approach, but put that thought on hold for the moment.