Doing the Wrong Problem PhL 11.25.13

Comments 12/5/13. Here is a summary of this doc:

• I consider the Helmholtz A_z problem with an assumed uniform J in the wire, but for arbitrary ω. I realize that this problem is "illogical" since the current cannot be uniform unless dramatic external means are used to maintain it so. This is why the title refers to this as "the wrong problem" if you are trying to get something that matches Chapter 2 where I treat "the right problem" -- the true wire problem.

• In Section 1 I go ahead and compute A_z using the Helmholtz integral, both inside and outside the wire, even though the problem is illogical. I make no mention of homo solutions and boundary conditions on A_z because at this time I thought the Helmholtz integral always gave the exact answer, just the way King presents it. This lack of homo solutions is probably OK if you assume $\mu_1 = \mu_2$.

• I then think maybe the low-ω limit of the above calculation is "logical" even though the general ω result is "illogical". If the low-ω limit is valid, then it ought to agree with the low-ω limit of my Chapter 2 calculation.

• So I then take the low-ω limit of the Chapter 2 in Section 3 and the low-ω limit of my Section 1 Helmholtz result in Section 4. I do find a certain amount of agreement, though constant terms in A_z do not match and probably they are not significant.

• Some fancy math is used in this doc, and for that reason alone I should keep this doc.

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0. Overview comments

I started out doing the low frequency problem in Appendix M (currently M)

$$(\nabla^2_{2D} + \beta_1^2)\, A_z(r) = -\,[I\mu_2/(\pi a^2)]\;\theta(r\leq a) \qquad \mathbf{B} = B_\theta\hat{\boldsymbol{\theta}} \;\text{ with }\; B_\theta = -\,\partial_r A_z\,. \qquad \text{(M.5.1)}$$

Here I am assuming a uniform current density! That is why this problem makes no sense for large ω.

Now in Chapter 2, I solve the problem I think for any ω, not just small ω, and the skin effect does come out of it.

Here I start with a forced uniform J_z and as ω gets large, the problem really makes no sense, because that is not how the current density ends up. You cannot prescribe it that way. [maybe you could with a set of filament current sources?]

Nevertheless, I did this problem as if the current density could be prescribed that way, and I did get a result. The result does not agree with Chapter 2 because there J does not end up uniform.

I want to record this calculation here because it involved doing some integrals that I may have to do again in some other context!

So here is that entire calculation which I will just store here:

1. Calculation of A_z for round wire with uniform-J using the Helmholtz integral.

Note: Here I just compute the Helmholtz integral. I don't consider boundary conditions on A_z and I therefore don't talk about homogeneous solutions that might be required.

Start with:

$$(\nabla^2_{2D} + \beta_1^2)\, A_z(r) = -\,[I\mu_2/(\pi a^2)]\;\theta(r\le a) \qquad \mathbf{B} = B_\theta\hat{\boldsymbol{\theta}} \quad \text{with} \quad B_\theta = -\,\partial_r A_z\,. \qquad \text{(M.5.1)}$$

The 2D free-space Helmholtz Green's function propagator is shown in (I.1.7) to be

$$g(x|x') = (j/4)\, H_0^{(1)}(\beta_1 R) \qquad R = R = |\mathbf{x}-\mathbf{x}'|$$

where $H_0^{(1)}$ is a Hankel function. Thus, we may write the particular solution to (M.5.1) as the following Helmholtz integral,

$$A_z^{H}(r) = \int dA'\,[(j/4)\, H_0^{(1)}(kR)\,]\;[I\mu_2/(\pi a^2)]\;\theta(r\le a) \qquad R = R = |\mathbf{x}-\mathbf{x}'|$$

$$= [I\mu_2/(\pi a^2)]\,(j/4)\int_0^a r'\,dr' \int_{-\pi}^{\pi} d\theta'\, H_0^{(1)}\!\left(\beta_1\sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')}\,\right)\,. \qquad \text{(M.5.2)}$$

$$= [I\mu_2/(\pi a^2)]\,(j/4)\int_0^a r'\,dr'\,[2I_3].$$

The angle integral may be evaluated as

$$I_3 = (1/2)\int_{-\pi}^{\pi} d\theta'\, H_0^{(1)}\!\left(\beta_1\sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')}\,\right)$$

$= \int_0^{\pi} dx\, H_0^{(1)}(\beta_1\sqrt{r^2+ r'^2- 2rr'\cos x})$

$= \int_0^{\pi} dx\, J_0(\beta_1\sqrt{r^2+ r'^2- 2rr'\cos x}) + j \int_0^{\pi} dx\, Y_0(\beta_1\sqrt{r^2+ r'^2- 2rr'\cos x})$

where in the last line we use this NIST fact,

10.4.3

$$H_\nu^{(1)}(z) = J_\nu(z) + i\,Y_\nu(z),$$
$$H_\nu^{(2)}(z) = J_\nu(z) - i\,Y_\nu(z),$$

Now try this

$\beta_1\sqrt{r^2+ r'^2- 2rr'\cos x} = \sqrt{(\beta_1 r)^2+ (\beta_1 r')^2- 2(\beta_1 r)(\beta_1 r')\cos x} = \sqrt{\alpha^2+ \beta^2- 2\alpha\beta\cos x}$

$\alpha = \beta_1 r \qquad \beta = \beta_1 r'$

Then we have

$I_3 = \int_0^{\pi} dx\, J_0(\sqrt{\alpha^2+ \beta^2- 2\alpha\beta\cos x}) + j \int_0^{\pi} dx\, Y_0(\sqrt{\alpha^2+ \beta^2- 2\alpha\beta\cos x})$

Then use GR7 page 726

6.684

1.[8]
$$\int_0^{\pi} (\sin x)^{2\nu} \frac{J_\nu\left(\sqrt{\alpha^2+\beta^2-2\alpha\beta\cos x}\right)}{\left(\sqrt{\alpha^2+\beta^2-2\alpha\beta\cos x}\right)^\nu} dx = 2^\nu\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)\frac{J_\nu(\alpha)}{\alpha^\nu}\frac{J_\nu(\beta)}{\beta^\nu}$$
$$\left[\operatorname{Re}\nu > -\tfrac{1}{2}\right] \qquad \text{ET II 362(27)}$$

2.
$$\int_0^{\pi} (\sin x)^{2\nu} \frac{Y_\nu\left(\sqrt{\alpha^2+\beta^2-2\alpha\beta\cos x}\right)}{\left(\sqrt{\alpha^2+\beta^2-2\alpha\beta\cos x}\right)^\nu} dx = 2^\nu\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)\frac{J_\nu(\alpha)}{\alpha^\nu}\frac{Y_\nu(\beta)}{\beta^\nu}$$
$$\left[|\alpha| < |\beta|, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right] \qquad \text{ET II 362(28)}$$

Set ν = 0 to get

$\int_0^{\pi} dx\, J_0(\beta_1\sqrt{r^2+ r'^2- 2rr'\cos x}) = \sqrt{\pi}\ \Gamma(1/2)\, J_0(\alpha)J_0(\beta) = \pi\, J_0(\alpha)J_0(\beta)$

$\int_0^{\pi} dx\, Y_0(\beta_1\sqrt{r^2+ r'^2- 2rr'\cos x}) = \sqrt{\pi}\ \Gamma(1/2)\, J_0(\alpha)Y_0(\beta) = \pi\, J_0(\alpha)Y_0(\beta)$ both for r < r'

Then

$I_3 = \pi\, J_0(\alpha)J_0(\beta) + j\,\pi\, J_0(\alpha)Y_0(\beta) = \pi\, J_0(\alpha)\,[J_0(\beta) + j\, Y_0(\beta)\,] = \pi\, J_0(\alpha)H_0^{(1)}(\beta)$

$= \pi\, J_0(\beta_1 r)H_0^{(1)}(\beta_1 r')$ for r < r' .

But I_3 is symmetric so then

$$I_3 = \pi J_0(\beta_1 r_<) H_0^{(1)}(\beta_1 r_>) = \pi J_0(\beta_1 r) H_0^{(1)}(\beta_1 r')\theta(r'>r) + \pi J_0(\beta_1 r') H_0^{(1)}(\beta_1 r)\theta(r'<r)$$

and then we have

$$A_z^H(r) = [I\mu_2/(\pi a^2)] (j/4) \int_0^a r' dr' [2I_3].$$

$$= [I\mu_2/(\pi a^2)] (j/4) 2 \int_0^a r' dr' \{ \pi J_0(\beta_1 r) H_0^{(1)}(\beta_1 r') \theta(r'>r) + \pi J_0(\beta_1 r') H_0^{(1)}(\beta_1 r)\theta(r'<r)\}$$

$$= [I\mu_2/(\pi a^2)](j/4)2\pi \{ J_0(\beta_1 r) \int_0^a r' dr' H_0^{(1)}(\beta_1 r') \theta(r'>r) + H_0^{(1)}(\beta_1 r) \int_0^a r' dr' J_0(\beta_1 r') \theta(r'<r) \}$$

The required integrals are

$$\int_0^a r' dr' H_0^{(1)}(\beta_1 r') \theta(r'>r) = \theta(a>r) \int_r^a r' dr' H_0^{(1)}(\beta_1 r')$$

$$= \theta(a>r)(1/\beta_1)^2 \int_{\beta_1 r}^{\beta_1 a} dx\, x H_0^{(1)}(x) = \theta(a>r)(1/\beta_1)^2 [x H_1^{(1)}(x)]_{\beta_1 r}^{\beta_1 a} \quad \text{GR7 p 630}$$

$$= \theta(a>r)(1/\beta_1)^2 [\beta_1 a H_1^{(1)}(\beta_1 a) - \beta_1 r H_1^{(1)}(\beta_1 r)]$$

$$= \theta(a>r)(1/\beta_1) [a H_1^{(1)}(\beta_1 a) - r H_1^{(1)}(\beta_1 r)] .$$

$$\int_0^a r' dr' J_0(\beta_1 r') \theta(r'<r) = \int_0^{\min(a,r)} r' dr' J_0(\beta_1 r') = (1/\beta_1)^2 \int_0^{\beta_1 \min(a,r)} dx\, x J_0(x)$$

$$= (1/\beta_1)^2 [x J_1(x)]_0^{\beta_1 \min(a,r)} = (1/\beta_1)^2 [\beta_1 \min(a,r) J_1(\beta_1 \min(a,r))]$$

$$= (1/\beta_1) \begin{cases} r J_1(\beta_1 r) & r<a \\ a J_1(\beta_1 a) & r>a \end{cases}$$

Then

$$A_z^H(r) = [I\mu_2/(\pi a^2)](j/4)2\pi (1/\beta_1) *$$
$$\{ J_0(\beta_1 r) \theta(a>r) [a H_1^{(1)}(\beta_1 a) - r H_1^{(1)}(\beta_1 r)] + H_0^{(1)}(\beta_1 r) \begin{cases} r J_1(\beta_1 r) & r<a \\ a J_1(\beta_1 a) & r>a \end{cases} \}$$

For the r<a region we then have

$$A_z^H(r) = [I\mu_2/(\pi a^2)](j/4)2\pi (1/\beta_1)$$
$$\{ J_0(\beta_1 r) [a H_1^{(1)}(\beta_1 a) - r H_1^{(1)}(\beta_1 r)] + H_0^{(1)}(\beta_1 r) r J_1(\beta_1 r) \} \qquad r<a$$

while for the r>a region,

$$A_z^H(r) = [I\mu_2/(\pi a^2)](j/4)2\pi\ (1/\beta_1)\ \{H_0^{(1)}(\beta_1 r)\ aJ_1(\beta_1 a)\} \qquad r > a$$

Do things match at r = a?

$$A_z^H(a) = [I\mu_2/(\pi a^2)](j/4)2\pi\ (1/\beta_1)$$
$$\{\ H_0^{(1)}(\beta_1 a)\ aJ_1(\beta_1 a)\ \} \qquad r<a$$

$$A_z^H(a) = [I\mu_2/(\pi a^2)](j/4)2\pi\ (1/\beta_1)\ \{H_0^{(1)}(\beta_1 a)\ aJ_1(\beta_1 a)\} \qquad r > a$$

Yes, they do match.

2. Calculate $A_z(r)$ implied by the Chapter 2 result.

Suppose I start with the known **Chapter 2** result,

$$B(r) = B(a)\ J_1(\beta_1 r) / J_1(\beta_1 a) \ = -\partial_r A_z(r)$$

$$\partial_r A_z(r) = -\ B(a)\ J_1(\beta_1 r) / J_1(\beta_1 a) \ = -k\ J_1(\beta_1 r) \qquad k = B(a)/\ J_1(\beta_1 a)$$

Then integrate

$$A_z(r) \ = -k \int^{r} J_1(\beta_1 r)dr \ = -k\ (1/\beta_1) \int^{r} J_1(\beta_1 r)d(\beta_1 r) = -k\ (1/\beta_1) \int^{\beta_1 r} J_1(x)\ dx$$

$$= \ + k\ (1/\beta_1)\ J_0(\beta_1 r) + \text{constant} \qquad r < a$$

$$= \ [B(a)/\ J_1(\beta_1 a)]\ (1/\beta_1)\ J_0(\beta_1 r) + \text{constant}$$

where I used this result from p 630 of GR7,

5.56

1. $$\int Z_1(x)\,dx = -\,Z_0(x)$$ JA

3. Question: what is the small-β_1 limit of this Chapter 2 expression?

$$J_1(\beta_1 a) \approx (\beta_1 a/2)$$
$$J_0(\beta_1 r) \approx 1$$

So then I get

$$A_z(r) \ = [B(a)/\ J_1(\beta_1 a)]\ (1/\beta_1)\ J_0(\beta_1 r) + \text{constant}$$

$\approx$ $[2B(a)/(\beta_1 a)]\ (1/\beta_1)\ 1$ + constant

$\approx$ $[2B(a)/(a)]\ (1/\beta_1^2)$ + constant

$B(a) = B_\theta(a) = \mu_2 I/(2\pi a)$ so

$A_z(r) \approx [\mu_2 I/(\pi a^2)]\ (1/\beta_1^2)$ + constant

What is my interpretation of this result? It is a very large constant term since $\beta_1 \rightarrow 0$. Now let's add another term and do it again

$J_1(\beta_1 a) \approx (\beta_1 a/2)$
$J_0(\beta_1 r) \approx 1 - (\beta_1 r/2)^2$

$A_z(r) = [B(a)/ J_1(\beta_1 a)]\ (1/\beta_1)\ J_0(\beta_1 r)$ + constant

$\approx$ $[2B(a)/(\beta_1 a)]\ (1/\beta_1)(\ 1 - (\beta_1 r/2)^2)$ + constant

$\approx$ $[2B(a)/(a)]\ (1/\beta_1^2)\ (\ 1 - (\beta_1 r/2)^2)$ + constant

$B(a) = B_\theta(a) = \mu_2 I/(2\pi a)$ so

$A_z(r) \approx [\mu_2 I/(\pi a^2)]\ (1/\beta_1^2)\ (\ 1 - (\beta_1 r/2)^2)$ + constant

$\approx$ $[\mu_2 I/(\pi a^2)]\ (1/\beta_1^2)$ - $[\mu_2 I/(2\pi a^2)]\ (1/\beta_1^2)\ (\beta_1 r/2)^2)$ + constant

$\approx$ $[\mu_2 I/(\pi a^2)]\ (1/\beta_1^2)$ - $[\mu_2 I/(2\pi a^2)]\ (r/2)^2$ + constant

$\approx$ $[\mu_2 I/(\pi a^2)]\ (1/\beta_1^2)$ - $[\mu_2 I/(2\pi a^2)]\ (r^2/4)$ + constant

$\approx$ - $[\mu_2 I/(\pi a^2)]\ (r^2/4)$ + constant'

The new term is *finite*, there is no smallness parameter multiplying it. and it is exactly the term which appears in my Poisson result (M.4.15) for r<a. So I am "happy" with this limit of the Chapter 2 result.

4. Calculate $A_z(r)$ from the Helmholtz integral in the limit of small β_1 for r < a

What does my new calculation say for small β_1 ?

$A_z^H(r) = [I\mu_2/(\pi a^2)](j/4)2\pi\ (1/\beta_1)$
$\{ J_0(\beta_1 r)\ [a\ H_1^{(1)}(\beta_1 a) - r\ H_1^{(1)}(\beta_1 r)\ ] + H_0^{(1)}(\beta_1 r)\ r\ J_1(\beta_1 r)\ \}$ r<a (***)

First, let's compute the small β_1 limit of [a $H_1^{(1)}(\beta_1 a)$ - r $H_1^{(1)}(\beta_1 r)$]. I will save time and use Maple to get a series for this thing for small x:

```
series(HankelH1(1,x),x,2);
```

$$-2\frac{I}{\pi}x^{-1}+\frac{1}{2}\frac{\pi+2I\ln(x)-2I\ln(2)+2I\gamma-I}{\pi}x+O(x^2)$$

So I read this as

$$H_1^{(1)}(x) = (-2j/\pi)(1/x) + (j/\pi)\, x \ln(x) + Ax + O(x^2) \qquad A = (1/2\pi)(\pi - 2j\ln(2) + 2j\gamma - j)$$

Then

$$x\, H_1^{(1)}(x) = (-2j/\pi) + (j/\pi)\, x^2 \ln(x) + Ax^2 + O(x^3)$$

Then

$$[x\, H_1^{(1)}(x) - y\, H_1^{(1)}(y)]$$

$$= [(-2j/\pi) + (j/\pi)\, x^2 \ln(x) + Ax^2 + O(x^3)] - [(-2j/\pi) + (j/\pi)\, y^2 \ln(y) + Ay^2 + O(y^3)]$$

$$= [(j/\pi)\, x^2 \ln(x) + Ax^2 + O(x^3)] - [(j/\pi)\, y^2 \ln(y) + Ay^2 + O(y^3)]$$

$$= (j/\pi)[x^2 \ln(x) - y^2 \ln(y)] + A(x^2 - y^2) + O(x^3) - O(y^3)$$

Now set $x = \beta_1 a$ and $y = \beta_1 r$ to get

$$[\beta_1 a\, H_1^{(1)}(\beta_1 a) - \beta_1 r\, H_1^{(1)}(\beta_1 r)]$$

$$= (j/\pi)[(\beta_1 a)^2 \ln(\beta_1 a) - (\beta_1 r)^2 \ln(\beta_1 r)] + A((\beta_1 a)^2 - (\beta_1 r)^2) + O((\beta_1 a)^3) - O((\beta_1 r)^3)$$

$$= \beta_1^2 \{ (j/\pi)[(a)^2 \ln(\beta_1 a) - (r)^2 \ln(\beta_1 r)] + A(a^2 - r^2) + O(\beta_1) \}$$

or

$$[a\, H_1^{(1)}(\beta_1 a) - r\, H_1^{(1)}(\beta_1 r)]$$
$$= \beta_1 ((j/\pi)[(a)^2 \ln(\beta_1 a) - (r)^2 \ln(\beta_1 r)] + A(a^2 - r^2) + O(\beta_1))$$

Now this entire object has smallness order β_1, keep that in mind. Look back now at

$$A_z^H(r) = [I\mu_2/(\pi a^2)](j/4)2\pi\ (1/\beta_1)$$
$$\{ J_0(\beta_1 r)\, [a\, H_1^{(1)}(\beta_1 a) - r\, H_1^{(1)}(\beta_1 r)] + H_0^{(1)}(\beta_1 r)\, r\, J_1(\beta_1 r) \} \qquad r<a \qquad (***)$$

Due to the $(1/\beta_1)$ factor, this object is going to contribute to the finite result! You cannot throw it out.

Now let's work on the small β_1 limit of second grouping $H_0^{(1)}(\beta_1 r)\, r\, J_1(\beta_1 r)$. I will invoke Maple for the first factor

```
series(HankelH1(0,x),x,2);
```

$$\frac{\pi + 2\,I\ln(x) - 2\,I\ln(2) + 2\,I\gamma}{\pi} + O(x^2)$$

which I interpret to say

$H_0^{(1)}(x) \approx (2j/\pi)\ln(x) + A' + O(x^2)$ $\qquad\qquad$ $A' = (1/\pi)(\pi - 2j\ln(2) + 2j\gamma)$

so

$H_0^{(1)}(\beta_1 r) \approx (2j/\pi)\ln(\beta_1 r) + A' + O((\beta_1 r)^2)$

The last term is order β_1^2 so I will likely ignore it. Next,

$J_1(x) \approx (x/2) - x^3/16$

$J_1(\beta_1 r) \approx (\beta_1 r/2) - (\beta_1 r)^3/16$

and I will surely drop this last term. Thus we have

$H_0^{(1)}(\beta_1 r)\, r\, J_1(\beta_1 r) \approx [(2j/\pi)\ln(\beta_1 r) + A' + O(\beta_1^2)]\;\; r\,[(\beta_1 r/2)]$

$= \beta_1 r^2/2\,[(2j/\pi)\ln(\beta_1 r) + A' + O(\beta_1^2)]]$

so I think I can drop the last term here and end up with

$H_0^{(1)}(\beta_1 r)\, r\, J_1(\beta_1 r) \approx \beta_1 (r^2/2)[(2j/\pi)\ln(\beta_1 r) + A']$

and again, this is an order β_1 term and it will contribute!

So now install these both into our $A_z(r)$ expression above which is for $r < a$

$A_z^H(r) = [I\mu_2/(\pi a^2)](j/4)2\pi\;(1/\beta_1)$
$\{ J_0(\beta_1 r)\,[a\, H_1^{(1)}(\beta_1 a) - r\, H_1^{(1)}(\beta_1 r)]$
$+\; H_0^{(1)}(\beta_1 r)\, r\, J_1(\beta_1 r)\,\}$

$A_z^H(r) = [I\mu_2/(\pi a^2)](j/4)2\pi\;(1/\beta_1)$
$\{ J_0(\beta_1 r)\,[\beta_1 (\;(j/\pi)[(a)^2\ln(\beta_1 a) - (r)^2\ln(\beta_1 r)] + A(a^2 - r^2) + O(\beta_1))]$
$+\; \beta_1 (r^2/2)[(2j/\pi)\ln(\beta_1 r) + A']\;\}$

$A_z^H(r) = [I\mu_2/(\pi a^2)](j/4)2\pi$
$\{ J_0(\beta_1 r)\,[\;(j/\pi)[(a)^2\ln(\beta_1 a) - (r)^2\ln(\beta_1 r)] + A(a^2 - r^2) + O(\beta_1)\;]$
$+\; (r^2/2)[(2j/\pi)\ln(\beta_1 r) + A']\;\}$

Now we don't want higher β_1 terms, so we now set $J_0(\beta_1 r) = 1$ and this then gives

$$A_z^H(r) = [I\mu_2/(\pi a^2)](j/4)2\pi$$
$$\{ [(j/\pi)[(a)^2 \ln(\beta_1 a) - (r)^2 \ln(\beta_1 r)] + A(a^2 - r^2)]$$
$$+ (r^2/2) [(2j/\pi) \ln(\beta_1 r) + A'] \}$$

$$A_z^H(r) = [I\mu_2/(\pi a^2)](j/4)2\pi$$
$$\{ [(j/\pi)[(a)^2 \ln(\beta_1 a) - (r)^2 \ln(\beta_1 r)] + A(a^2 - r^2)] + (r^2/2) [(2j/\pi) \ln(\beta_1 r) + A'] \}$$

$$A_z^H(r) = [I\mu_2/(\pi a^2)](j/4)2\pi$$
$$\{ [(j/\pi)[(a)^2 \ln(\beta_1 a) - (r)^2 \ln(\beta_1 r)] + A(a^2 - r^2)] + r^2 [(j/\pi) \ln(\beta_1 r) + A' /2] \}$$

$$A_z^H(r) = [I\mu_2/(\pi a^2)](j/4)2\pi$$
$$\{ (j/\pi)a^2 \ln(\beta_1 a) - (j/\pi) r^2 \ln(\beta_1 r) + A(a^2 - r^2) + r^2 (j/\pi) \ln(\beta_1 r) + A' (r^2/2) \}$$

$$A_z^H(r) = [I\mu_2/(\pi a^2)](j/4)2\pi$$
$$\{ (j/\pi)a^2 \ln(\beta_1 a) + A(a^2 - r^2) + A' (r^2/2) \}$$

$$A_z^H(r) = [I\mu_2/(\pi a^2)](j/4)2\pi$$
$$\{ (j/\pi)a^2 \ln(\beta_1 a) + Aa^2 - Ar^2 + (A'/2) r^2 \}$$

$$A_z^H(r) = [I\mu_2/(\pi a^2)](j/4)2\pi$$
$$\{ (j/\pi)a^2 \ln(\beta_1 a) + Aa^2 + r^2 (A'/2 - A) \}$$

Now go back to

$$A' = (1/\pi)(\pi - 2j \ln(2) + 2j\gamma)$$
$$A = (1/2\pi) (\pi - 2j \ln(2) + 2j\gamma - j)$$

or

$$A'/2 = (1/2\pi)(\pi - 2j \ln(2) + 2j\gamma)$$
$$A = (1/2\pi) (\pi - 2j \ln(2) + 2j\gamma - j)$$

Then

$$(A'/2 - A) = (1/2\pi)(\pi - 2j \ln(2) + 2j\gamma) - (1/2\pi) (\pi - 2j \ln(2) + 2j\gamma - j)$$

$$= (1/2\pi) [(\pi - 2j \ln(2) + 2j\gamma) - (\pi - 2j \ln(2) + 2j\gamma - j)]$$

$$= (1/2\pi) [\pi - 2j \ln(2) + 2j\gamma - \pi + 2j \ln(2) - 2j\gamma + j]$$

$$= (1/2\pi) [+ j] = (j/2\pi)$$

Then we have

$$A_z^H(r) = [I\mu_2/(\pi a^2)](j/4)2\pi \{ (j/\pi)a^2 \ln(\beta_1 a) + Aa^2 + r^2 (A'/2 - A) \}$$

$$A_z^H(r) = [I\mu_2/(\pi a^2)](j/4)2\pi \{ (j/\pi)a^2 \ln(\beta_1 a) + Aa^2 + r^2 (j/2\pi) \}$$

$$A_z^H(r) = [I\mu_2/(\pi a^2)](j/4)2\pi \{ (j/\pi)a^2 \ln(\beta_1 a) + Aa^2 + r^2 (j/2\pi) \}$$

$$= [I\mu_2/(\pi a^2)](j/4)2\pi \{(j/\pi)a^2 \ln(\beta_1 a) + Aa^2\} + [I\mu_2/(\pi a^2)](j/4)2\pi \{r^2 (j/2\pi)\}$$

$$= [I\mu_2/(\pi)](j/2) \{(j) \ln(\beta_1 a) + A\} + [I\mu_2/(\pi a^2)](j/4) \{r^2 (j)\}$$

$$= [I\mu_2/(\pi)](j/2) \{(j) \ln(\beta_1 a) + A\} - [I\mu_2/(\pi a^2)](r^2/4)$$

$$= - [I\mu_2/(\pi a^2)](r^2/4) + [I\mu_2/(\pi)](j/2) \{(j) \ln(\beta_1 a) + A\}$$

$$= - [I\mu_2/(\pi a^2)](r^2/4) + [I\mu_2/(\pi)](j/2) \{(j) \ln(\beta_1 a) + A\}$$

As a reminder, here was the Chapter 2 result calculated above

$$A_z(r) \approx - [\mu_2 I/(\pi a^2)] (r^2/4) + \text{constant}'$$

Absolutely amazingly, these two terms have the exact same form in terms of the r^2 non constant piece. This was the result I have spent maybe 3 hours trying to get. Everything is right including the sign.

Comment: In this limit, the Helmholtz integral result has a very strange constant term which includes the γ constant!

Question: Do I already know the B solution for Chapter 2 *outside* the wire? Can I just use Ampere's Law, or do I have to worry about $j\omega D$? Carry on here tomorrow!

********** I will not use this but save**********

The next task is to compute the B field using $B_\theta = - \partial_r A_z$. We first note that

$$\partial_r [r J_1(\beta_1 r)] = r \partial_r J_1(\beta_1 r) + J_1(\beta_1 r) = r \beta_1 J_1'(\beta_1 r) + J_1(\beta_1 r)$$

$$\partial_r [r H_1^{(1)}(\beta_1 r)] = r \partial_r H_1^{(1)}(\beta_1 r) + H_1^{(1)}(\beta_1 r) = r \beta_1 H_1^{(1)}'(\beta_1 r) + H_1^{(1)}(\beta_1 r)$$

Then for $r < a$,

$$-B_\theta = \partial_r A_z = C \beta_1 J_0'(\beta_1 r) [a H_1^{(1)}(\beta_1 a) - r H_1^{(1)}(\beta_1 r)] + C J_0(\beta_1 r) [-\{ r \beta_1 H_1^{(1)}'(\beta_1 r) + H_1^{(1)}(\beta_1 r)\}]$$

$$-B_\theta = \partial_r A_z = -C\,\beta_1 J_1(\beta_1 r)\,[a\,H_1^{(1)}(\beta_1 a) - r\,H_1^{(1)}(\beta_1 r)] + C\,J_0(\beta_1 r)\,[-\{r\,\beta_1\,H_1^{(1)}{}'(\beta_1 r) + H_1^{(1)}(\beta_1 r)\}] \qquad r < a$$

and for r > a

$$-B_\theta = \partial_r A_z = C\,\beta_1\,H_0^{(1)}{}'(\beta_1 r)\,[a\,J_1(\beta_1 a) - r\,J_1(\beta_1 r)] + C\,H_0^{(1)}(\beta_1 r)\,[-\{r\,\beta_1\,J_1'(\beta_1 r) + J_1(\beta_1 r)\}] \qquad r > a$$

Now the $64 question: I computed B_θ for r < a in Chapter 2 and got this result

$$B_\theta(r) = +B(a)\,[J_1(\beta_1 r) / J_1(\beta_1 a)] \qquad r<a$$

Why are these results so different looking? Well, as noted at the start, the problem I solved here is an unphysical problem because it assumes a constant current density, but in reality we know there will be skin effect and a non-uniform J_z therefore!

BUT, the results should agree for small β_1 and they don't agree!

Suppose I start with the known Chapter 2 result,

$$B_\theta(r) = k\,J_1(\beta_1 r) = -\partial_r A_z(r)$$

Then integrate

$$A_z(r) = -k\int^r J_1(\beta_1 r)dr = -k\,(1/\beta_1)\int^r J_1(\beta_1 r)d(\beta_1 r) = -k\,(1/\beta_1)\int^{\beta_1 r} J_1(x)\,dx$$

$$= +k\,(1/\beta_1)\,J_0(\beta_1 r) + \text{constant}$$

where I used this result from p 630 of GR7,

5.56

1. $\int Z_1(x)\,dx = -Z_0(x)$ JA

Now compare this with my r < a result above

$$A_z^H(r) = C\,J_0(\beta_1 r)\,[a\,H_1^{(1)}(\beta_1 a) - r\,H_1^{(1)}(\beta_1 r)] \qquad r < a$$

For small β_1, this should equal what I just did above, but it does not work due to the bracket factor.

For small β_1 we in fact know that

10.7.7

$$H_\nu^{(1)}(z) \sim -H_\nu^{(2)}(z) \sim -(i/\pi)\,\Gamma(\nu)(\tfrac{1}{2}z)^{-\nu}, \quad \Re\nu > 0.$$

so

$H_1^{(1)}(z) \approx -(j/\pi)(2/z)$ => $z\,H_1^{(1)}(z) \approx -(j/\pi)(2) = (-2j/\pi)$

Then we get [...] ≈ 0.

So why does my full solution have the wrong small-β_1 limit?