Old App G no longer used as support for Chapter 4 PhL 1.17.14

Appendix G: Chapter 4 Support

This Appendix provides technical support for certain claims made in Chapter 4.

G.1 Evaluation of the integral (4.1.9)

(a) The first claim is that the integral on the left below has the form shown on the right:

$$I_n(\mathbf{x}) = \int_{-\infty}^{\infty} dz' \, (z'-z)^n \frac{e^{-j\beta R}}{R} = s^n f_n(\beta s) \qquad \text{(G.1)}$$

where

$$R = \sqrt{(z'-z)^2 + s^2} \quad \text{and} \quad s = \sqrt{(x-x')^2 + (y-y')^2} \quad .$$

Here $I_n(\mathbf{x}) = I(x,y,z)$ is just our name for this integral, please do not confuse this with the modified Bessel function $I_n(x)$.

To show the claim, the first step is to define $z'' \equiv z'-z$ to get

$$I_n(\mathbf{x}) = \int_{-\infty}^{\infty} dz'' \, (z'')^n \frac{e^{-j\beta R}}{R} \qquad R = \sqrt{(z'')^2 + s^2} \qquad \text{(G.2)}$$

which shows that $I(\mathbf{x})$ is independent of z so we can think of (G.2) as (G.1) with z = 0.

Next define $x \equiv z''/s$ to get

$$dz'' \, (z'')^n = s^{n+1} \, dx \, x^n$$

$$R = \sqrt{(xs)^2 + s^2} = s\sqrt{x^2 + 1}$$

so that

$$I_n(\mathbf{x}) = s^n \int_{-\infty}^{\infty} dx \, x^n \frac{\exp(-js\beta\sqrt{x^2+1})}{\sqrt{x^2+1}} = s^n f_n(s\beta)$$

which then confirms the claimed form in (G.1).

(b) Our next task is then to compute this integral

$$f_n(\alpha) = \int_{-\infty}^{\infty} dx \, x^n \frac{\exp(-j\alpha\sqrt{x^2+1})}{\sqrt{x^2+1}} \quad . \qquad \alpha \equiv s\beta$$

For odd integer n, the integral clearly vanishes since the integrand is then an odd function of x. For even n we replace the integral with double its positive-side value and replace n = 2m with m = 1,2,3 ... to list off the non-vanishing integrals :

$$f_{2m}(\alpha) = 2\int_0^\infty dx\, x^{2m}\frac{\exp(-j\alpha\sqrt{x^2+1})}{\sqrt{x^2+1}} \quad . \qquad m = 0,1,2... \qquad (G.3)$$

Finally, change to $y = \sqrt{x^2+1}$ which says $x^2 = y^2 - 1$ and so xdx = ydy. Then

$$dx\, x^{2m} = (y/x)dy\, x^{2m} = y\, dy\, x^{2m-1} = y\, dy\, (y^2-1)^{m-1/2}$$

and so

$$f_{2m}(\alpha) = 2\int_1^\infty dy\,(y^2-1)^{m-1/2}\, e^{-j\alpha y} \quad . \qquad (G.4)$$

Finally we have something we can find in the standard tables. GR7 3.387 3 (p 350) claims,

3. $$\int_1^\infty (x^2-1)^{\nu-1} e^{-\mu x}\, dx = \frac{1}{\sqrt{\pi}}\left(\frac{2}{\mu}\right)^{\nu-\frac{1}{2}} \Gamma(\nu)\, K_{\nu-\frac{1}{2}}(\mu)$$

$$\left[|\arg\mu| < \frac{\pi}{2}, \quad \mathrm{Re}\,\nu > 0\right] \qquad \text{WA 190(4)a}$$

with $\nu = m+1/2$ and $\mu = j\alpha = js\beta$. Recall from (1.5.1) that $\beta^2 = \mu\varepsilon\omega^2 - j\omega\mu\sigma$ so β^2 and therefore β has a small negative imaginary part causing $\mu = js\beta$ to have a small positive real part which then allows the integration to converge due to the $e^{-\mu x}$ factor. Therefore,

$$f_{2m}(\alpha) = (2/\sqrt{\pi}\,)\,(2/j\alpha)^m\,\Gamma(m+1/2)\,K_m(j\alpha) \qquad (G.5)$$

$$= (2^{m+1}/\sqrt{\pi}\,)\,\Gamma(m+1/2)\,K_m(j\alpha)\,/\,(j\alpha)^m$$

where $K_m(z)$ is a modifed Bessel function.

For m = 0 one has $\Gamma(m+1/2) = \sqrt{\pi}$. For m = 1,2,3 one can write

$$\Gamma(m+1/2) = (2m-1)!!\,\sqrt{\pi}\,/2^m \qquad \text{Spiegel 16.6} \qquad (G.6)$$

where for example 5!! = 5*3*1. These can be combined into a single formula if one interprets (-1)!! = 1 as well as (0)!! = 1. Then,

$$f_{2m}(\alpha) = (2^{m+1}/\sqrt{\pi}\,)\,[(2m-1)!!\,\sqrt{\pi}\,/2^m]\,K_m(j\alpha)\,/\,(j\alpha)^m$$

$$= 2\,(2m-1)!!\,K_m(j\alpha)\,/\,(j\alpha)^m \qquad (G.7)$$

and then restoring 2m = n we have

$f_n(\alpha) = 2\,(n-1)!!\; K_{n/2}(j\alpha) / (j\alpha)^{n/2}$ for n even

$f_n(\alpha) = 0$. for n odd (G.8)

Using then the fact that

$$s^n f_n(\beta s) = s^n\, 2\,(n-1)!!\; K_{n/2}(j\beta s) / (j\beta s)^{n/2} \tag{G.9}$$

we conclude that

$$I_n(\mathbf{x}) = \int_{-\infty}^{\infty} dz'\,(z'-z)^n \frac{e^{-j\beta R}}{R}$$

$$= s^n f_n(\beta s) = \begin{cases} 0 & \text{for odd n} \\ 2s^n\,(n-1)!!\; K_{n/2}(j\beta s) / (j\beta s)^{n/2} & \text{for n even} \end{cases} \tag{G.10}$$

In particular,

$$I_0(\mathbf{x}) = \int_{-\infty}^{\infty} dz' \frac{e^{-j\beta R}}{R} = 2\,K_0(j\beta s)\,. \tag{G.11}$$

The modified Bessel function $K_m(z)$ is described for example in NIST Chapter 10 p 248 and elsewhere. For small z, we find this useful series expansion for $K_m(z)$ in GR7 p 919,

$$\mathbf{8.446}^8 \quad K_n(z) = \frac{1}{2}\sum_{k=0}^{n-1}(-1)^k \frac{(n-k-1)!}{k!\left(\frac{z}{2}\right)^{n-2k}}$$

$$+(-1)^{n+1}\sum_{k=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{n+2k}}{k!(n+k)!}\left[\ln\frac{z}{2} - \frac{1}{2}\psi(k+1) - \frac{1}{2}\psi(n+k+1)\right]$$

WA 95(15)

which we rewrite as,

$K_0(z) \approx -\ln(z/2)\,[\,1 + (z/2)^2 + O(z^4)\,] + \psi(1) + (z/2)^2\,\psi(2) + O(z^4)$ m = 0

$K_m(z) \approx 2^{m-1}(m-1)!\; z^{-m} - 2^{m-3}\,(m-2)!\; z^{-m+2} + O(z^{-m+4})$. m = 1,2,3... (G.12)

Here $\psi(z) \equiv \partial_z(\ln\Gamma(z))$ is the "digamma function", and $\psi(1)$ and $\psi(2)$ are just certain constants,

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evalf(Psi(1));
          -.5772156649
evalf(Psi(2));
           .4227843351
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The leading terms in the above small-z expansions are

$K_0(z) \approx -\ln(z/2)$ m = 0

$$K_m(z) \approx 2^{m-1}(m-1)!\, z^{-m}. \qquad m = 1,2,3... \qquad \text{(G.13)}$$

Notice that K_0 is fundamentally different from the other K_m in this limit: $K_0(z)$ is logarithmically divergent at z = 0, whereas $K_m(z)$ diverges as a power z^{-m} for m > 0.

G.2 Examination of higher order terms in (4.2.2)

Equation (4.2.2) reads,

$$V(z) = \frac{1}{4\pi\xi} \sum_{m=0}^{\infty} \frac{2\,(2m-1)!!}{(2m)!} \frac{1}{(j\beta)^m} q^{(2m)}(z) *$$
$$\{ \int_{C_1} dx'\, dy'\, a_1(x',y') [\, s_1^m K_m(j\beta s_1) - s_2^m K_m(j\beta s_2)]$$
$$- \int_{C_2} dx'\, dy'\, a_2(x',y') [\, s_1^m K_m(j\beta s_1) - s_2^m K_m(j\beta s_2)] \ \} \qquad \text{(G.14)}$$

We wish to show that in the small β limit, all terms with m= 1,2,3... can be neglected. Consider these factors which contain all the β dependence in the above expression

$$(j\beta)^{-m} [\, s_1^m K_m(j\beta s_1) - s_2^m K_m(j\beta s_2)] .$$

For m = 0 the leading terms for small β give

$$[K_0(j\beta s_1) - K_0(j\beta s_2)] \approx -\ln((j\beta s_1/2) [\, 1 + ((j\beta s_1/2)^2] + \ln((j\beta s_2/2) [\, 1 + ((j\beta s_2/2)^2]$$

$$= -\ln((j\beta s_1/2) [\, 1 + O(\beta^2)] + \ln((j\beta s_2/2) [\, 1 + O(\beta^2)]$$

$$= -\{\ln(\beta) + \ln(js_1/2)\} [\, 1 + O(\beta^2)] + \{\ln(\beta) + \ln(js_2/2)\} [\, 1 + O(\beta^2)]$$

$$= \ln(s_2/s_1) + O(\beta^2) .$$

For m > 0, this same factor behaves as

$$[K_m(j\beta s_1) - K_m(j\beta s_2)]$$

$$= \{2^{m-1}(m-1)!\,(j\beta s_1)^{-m} - 2^{m-3}(m-2)!\,(j\beta s_1)^{-m+2} + O((j\beta s_1)^{-m+4})\}$$
$$- \{2^{m-1}(m-1)!\,(j\beta s_2)^{-m} - 2^{m-3}(m-2)!\,(j\beta s_2)^{-m+2} + O((j\beta s_2)^{-m+4})\}$$

$$= (j\beta s_1)^{-m} \{2^{m-1}(m-1)! - 2^{m-3}(m-2)!\,(j\beta s_1)^2 + O((j\beta s_1)^4)\}$$
$$- (j\beta s_2)^{-m} \{2^{m-1}(m-1)! - 2^{m-3}(m-2)!\,(j\beta s_2)^2 + O((j\beta s_2)^4)\}$$

$$= (j\beta s_1)^{-m} \{2^{m-1}(m-1)! + O(\beta^2)\}$$
$$- (j\beta s_2)^{-m} \{2^{m-1}(m-1)! + O(\beta^2)\}$$

$$= (j\beta)^{-m}\ 2^{m-1}(m-1)!\ \{\ s_1^{-m} - s_2^{-m}\ +\ O(\beta^2)\ \}$$

Thus we compare the factors as they appear in (G.14)

$$\frac{1}{(j\beta)^0}\ [\ \ln(s_2/s_1) + O(\beta^2)] \qquad\qquad m = 0$$

$$\frac{1}{(j\beta)^m}\ 2^{m-1}(m-1)!\ (j\beta)^{-m}\ \{\ s_1^{-m} - s_2^{-m}\ + O(\beta^2)\ \}$$

$$= \frac{1}{(j\beta)^{2m}}\ 2^{m-1}(m-1)!\ \{\ s_1^{-m} - s_2^{-m}\ + O(\beta^2)\ \} \qquad\qquad m = 1,2,3...$$

The m = 0 term is seen to be order $\beta^0 = 1$ while all other terms are order β

$$V(z) = \frac{1}{4\pi\xi} \sum_{m=0}^{\infty} \frac{2\ (2m-1)!!}{(2m)!}\ \frac{1}{(j\beta)^m}\ q^{(2m)}(z)\ *$$
$$\{\ \int_{C_1} dx'\ dy'\ a_1(x',y')\ [\ s_1^m\ K_m\ (j\beta s_1) - s_2^m\ K_m\ (j\beta s_2)]$$
$$-\ \int_{C_2} dx'\ dy'\ a_2(x',y')\ [\ s_1^m\ K_m\ (j\beta s_1) - s_2^m\ K_m\ (j\beta s_2)]\ \ \}$$ (G.14)

Inserting the above small argument expansion for $K_m(z)$ we find

$$(j\beta)^{-m}\ [\ s_1^m\ K_m\ (j\beta s_1)\ - s_2^m\ K_m\ (j\beta s_2)]\ =$$

$$(j\beta)^{-m}\ [\ s_1^m\ \{\ 2^{m-1}(m-1)!\ (j\beta s_1)^{-m}\ - 2^{m-3}\ (m-2)!\ (j\beta s_1)^{-m+2} + O(\ (j\beta s_1)^{-m+4})\}$$
$$- s_2^m\ \{\ 2^{m-1}(m-1)!\ (j\beta s_2)^{-m}\ - 2^{m-3}\ (m-2)!\ (j\beta s_2)^{-m+2} + O(\ (j\beta s_2)^{-m+4})\}\ ]$$

Let us insert the small z expansion (G.12) for $K_m(z)$ into the square bracketed factors in (G.14). We get:

$$(j\beta)^{-m}\ [\ s_1^m\ K_m\ (j\beta s_1)\ - s_2^m\ K_m\ (j\beta s_2)] =$$

$$+\ 2^{m-1}(m-1)!\ [\ 1 - 1\ ] \qquad\qquad /\text{leading term}$$

$$-\ 2^{m-3}\ (m-2)!\ (j\beta)^2\ [\ s_1^2 - s_2^2\ ] \qquad\qquad /\ \text{second term, order}(\beta^2)$$

$$+ 2^{m-5}\,(m-3)!\,(j\beta)^4\,[\,s_1^4 - s_2^4\,] \qquad \text{/ third term, order}(\beta^4)$$

$$+ \ldots$$

The leading term, which would have been significant, completely cancels. The remaining terms are of order β^2 and smaller. For example, the second term has a coefficient which contains β^2 and involves finite integrals over the charge density of the following form:

$$\int_{C_1} dx'\,dy'\,a_1(x',y')\,[\,s_1^2 - s_2^2\,]$$

Thus, each term in the sum over m in (9) is of order β^2 or less. That is, it might happen that the integral above vanishes, so then each term would be of order β^4. Since $\beta = 2\pi/\lambda$ is a smallness parameter, we conclude that only the m=0 term is significant. This term was analyzed in the text of Chapter 4.