Retiring old section G.4 PhL 2.7.14

Here I had my "double" evaluation of the integral here called $I_1(r,r')$ which I realized is the same as Q from Appendix B. So here is the old section just for the record.

G.4 Solution for A_z using the 2D Helmholtz Integral

This method is technically more difficult than the first two methods shown in Section G.2 (solving the ODE and adding a homogeneous solution to meet the boundary conditions) and Section G.3 (instant Ampere's Law solution), so we show a lot of detail. The method is important because our entire Chapter 4 is based on using Helmholtz integrals to develop the theory of transmission line parameters, and this is one Helmholtz integral that we can actually compute without *too* much effort. An important result we find is that, when $\mu_1 \neq \mu_2$, the Helmholtz integral by itself does not supply the complete solution, and one must add in some amount of homogeneous solution of $\nabla^2_{2D} A_z(r) = 0$ in order to meet the required boundary conditions at r = a.

We really have a Poisson integral since $\beta_1^2 = 0$, but the full Helmholtz integral works the same way so we keep referring to it as a Helmholtz integral.

Recall from above:

$$\nabla^2_{2D} A_z(r) = - [I\mu_2/(\pi a^2)]\ \theta(r \leq a) \qquad \mathbf{B} = B_\theta \hat{\boldsymbol{\theta}} \quad \text{with} \quad B_\theta = - \partial_r A_z \,. \qquad (G.1.7)$$

Using the 2D free-space Green's function (propagator) as reviewed in Appendix I equation (I.1.6),

$$g(x|x') = (1/2\pi) \ln(1/R) = - (1/4\pi) \ln(R^2) \qquad R = R = |\mathbf{x}-\mathbf{x}'| \qquad (G.4.1)$$

we may write the particular solution to (G.1.7) as the following "Helmholtz" integral [see (I.1.8)]

$$A_z^H(r) = - \int dA' \ [(1/4\pi) \ln(R^2)\ ]\ [I\mu_2/(\pi a^2)]\ \theta(r \leq a)$$

$$= - \int_0^a r'\, dr' \int_{-\pi}^{\pi} d\theta' \ [(1/4\pi) \ln(r^2 + r'^2 - 2rr'\cos(\theta-\theta'))]\ [I\mu_2/(\pi a^2)]$$

$$= - (1/4\pi)\ [I\mu_2/(\pi a^2)] \int_0^a r'\, dr' \left[\int_{-\pi}^{\pi} d\theta' \ln(r^2 + r'^2 - 2rr'\cos(\theta-\theta')) \right]$$

$$= - (1/4\pi)\ [I\mu_2/(\pi a^2)] \int_0^a r'\, dr' \left[2\, I_1(r,r') \right] \qquad (G.4.2)$$

where

$$I_1(r,r') \equiv (1/2) \int_{-\pi}^{\pi} d\theta' \ln(r^2 + r'^2 - 2rr'\cos(\theta-\theta')) = (1/2) \int_{-\pi}^{\pi} d\theta'' \ln(r^2 + r'^2 - 2rr'\cos(\theta''))$$

$$= \int_0^{\pi} dx \ln(r^2 + r'^2 - 2rr'\cos x) . \qquad \text{(G.4.3)}$$

Since this integral I_1 is quite important, we evaluate it two independent ways:

Method 1:

$$I_1 = \int_0^{\pi} dx \ln[r^2(1 + r'^2/r^2 - 2r'/r\cos x)]$$

$$= \int_0^{\pi} dx \ln[r^2(1 + \alpha^2 - 2\alpha\cos x)] \qquad \alpha \equiv r'/r$$

$$= \int_0^{\pi} dx \ln[r^2] + \int_0^{\pi} dx \ln(1 + \alpha^2 - 2\alpha\cos x)$$

$$= \pi \ln(r^2) + I_2 . \qquad \text{(G.4.4)}$$

We then use this easily derived expansion Stakgold Vol II p 104 (6.22),

$$\ln(1 + \alpha^2 - 2\alpha\cos x) = -2 \sum_{n=1}^{\infty} (\alpha^n/n)\cos(nx) \qquad |\alpha| < 1 \qquad \text{(G.4.5)}$$

which is just a Fourier Cosine Series expansion of $\ln(1 + \alpha^2 - 2\alpha\cos x)$. Then,

$$I_2 = \int_0^{\pi} dx \ln(1 + \alpha^2 - 2\alpha\cos x) = -2 \sum_{n=1}^{\infty} (\alpha^n/n) \int_0^{\pi} dx \cos(nx)$$

$$= -2 \sum_{n=1}^{\infty} (\alpha^n/n) [(1/n)\sin(nx)]|_0^{\pi} = -2 \sum_{n=1}^{\infty} (\alpha^n/n^2) [\sin(n\pi) - \sin(0)] = 0$$

and then

$$I_1 = \pi \ln(r^2) + I_2 = \pi \ln(r^2) + 0 = 2\pi \ln(r) \qquad \alpha < 1 \quad \Leftrightarrow \quad r' < r$$

Since $I_1(r,r')$ is symmetric under $r \leftrightarrow r'$ we must have

$$I_1 = 2\pi \begin{cases} \ln(r) & r' < r \\ \ln(r') & r < r' \end{cases} = 2\pi \begin{cases} \ln(r) & r' < r \\ \ln(r') & r' > r \end{cases} . \qquad \text{(G.4.6)}$$

Method 2:

$$I_1 = \int_0^{\pi} dx \ln(r^2 + r'^2 - 2rr'\cos x) = \int_0^{\pi} dx \ln(a + b\cos x) .$$

This integral may be evaluated using GR7 p 531 4.224,

$$9.^8 \qquad \int_0^{\pi} \ln(a + b\cos x)\, dx = \pi \ln \frac{a + \sqrt{a^2 - b^2}}{2} \qquad [a \geq |b| > 0] \qquad \text{GW (322)(15)}$$

with $a = r'^2 + r^2$ and $b = -2rr'$ and $a^2 - b^2 = (r'^2 - r^2)^2$ so that $\sqrt{a^2 - b^2} = |\, r'^2 - r^2 \,|$. The condition $a > |b| > 0$ is met since $(r \pm r')^2 > 0 \Rightarrow r^2 + r'^2 > \pm 2rr'$ which says $a > \pm b$ so $a > |b|$. Thus,

$$I_1 = \int_0^{\pi} dx \ln [r'^2 + r^2 - 2rr' \cos x)] = \pi \ln \left[\frac{(r'^2 + r^2) + |\, r'^2 - r^2 \,|}{2} \right] = \begin{cases} \pi \ln [r'^2] & r' > r \\ \pi \ln [r^2] & r' < r \end{cases}$$

$$= 2\pi \begin{cases} \ln(r) & r' < r \\ \ln(r') & r' > r \end{cases} \qquad \text{(G.4.7)}$$

which agrees with the result of method 1.

Using the notation $\theta(x > a) \equiv \theta(x - a)$ [Heaviside step] we write (G.4.7) as

$$I_1 = 2\pi \{ \ln(r)\theta(r' < r) + \ln(r')\, \theta(r' > r) \} . \qquad \text{(G.4.8)}$$

Note: We use notation $\theta(x > a)$ in place of the more correct $\theta(x - a)$ because it requires "less thinking" to interpret.

Our particular "Helmholtz" integral (G.4.2) is then

$$A_z^H(r) = -(1/4\pi)\, I\mu_2/(\pi a^2) \int_0^a r'\, dr' [\, 2\, I_1(r,r') \,]$$

$$= -(1/4\pi)\, I\mu_2/(\pi a^2) \int_0^a r'\, dr'\ 4\pi \{ \ln(r)\theta(r' < r) + \ln(r')\, \theta(r' > r) \}$$

$$= -I\mu_2/(\pi a^2) \int_0^a r'\, dr' \{ \ln(r)\theta(r' < r) + \ln(r')\, \theta(r' > r) \}$$

$$= -I\mu_2/(\pi a^2) \{ \ln(r) \int_0^a r'\, dr'\, \theta(r' < r) + \int_0^a r'\, dr'\, \ln(r')\, \theta(r' > r) \} . \qquad \text{(G.4.9)}$$

The two integrals are

$$\int_0^a r'\, dr'\, \theta(r' < r) = \int_0^{\min(a,r)} r'\, dr' = (1/2)\, [\min(a,r)]^2 \qquad \text{(G.4.10)}$$

$$\int_0^a r'\, dr'\, \ln(r')\, \theta(r' > r) = \theta(r < a) \int_r^a r'\, dr'\, \ln(r') = \theta(r < a) \int_r^a x \ln x\, dx$$

$$= \theta(r<a)\,[(1/4)\,(r^2-a^2) + (1/2)\,a^2\ln a - (1/2)\,r^2\ln r\,] \quad . \qquad \text{(G.4.11)}$$

Then

$$A_z^H(r) = -\,I\mu_2/(\pi a^2)\,\{\,\ln(r)\int_0^a r'\,dr'\,\theta(r'<r) + \int_0^a r'\,dr'\,\ln(r')\,\theta(r'>r)\,\}$$

$$= -\,I\mu_2/(\pi a^2)\,\{\,\ln(r)\,(1/2)\,[\min(a,r)]^2 + \theta(r<a)\,[(1/4)\,(r^2-a^2) + (1/2)\,a^2\ln a - (1/2)\,r^2\ln r\,]\,\}\ .$$

For r > a we get

$$A_z^H(r) = -\,I\mu_2/(\pi a^2)\,\{\,\ln(r)\,(1/2)a^2\,\} = -\,I\mu_2/(2\pi)\,\ln(r)$$

$$\partial_r A_z^H(r) = -\,I\mu_2/(2\pi)\,(1/r)\ . \qquad \text{(G.4.12)}$$

For r < a the result is

$$A_z^H(r) = -\,I\mu_2/(\pi a^2)\,\{\,\ln(r)\,(1/2)r^2 + (1/4)\,(r^2-a^2) + (1/2)\,a^2\ln a - (1/2)\,r^2\ln r\,\}$$

$$= -\,I\mu_2/(\pi a^2)\,\{\,(1/4)\,(r^2-a^2) + (1/2)\,a^2\ln a\,\}$$

$$\partial_r A_z^H(r) = -\,I\mu_2/(\pi a^2)\,(r/2)\ . \qquad \text{(G.4.13)}$$

Recall the two boundary conditions

$$[A_z(a)]^1 = [A_z(a)]^2\ .$$
$$(1/\mu_1)\,[(\partial_r A_z)(a)]^1 = (1/\mu_2)\,[(\partial_r A_z)(a)]^2\ . \qquad \textit{(G.2.6)}$$

For our particular Helmholtz integral $A_z^H(r)$ we evaluate these boundary conditions to find

$$-\,I\mu_2/(2\pi)\,\ln(a) = -\,I\mu_2/(\pi a^2)\,(1/2)\,a^2\ln a$$
$$(1/\mu_1)\,[-\,I\mu_2/(2\pi)\,(1/a)] = (1/\mu_2)\,[-\,I\mu_2/(\pi a^2)\,(a/2)]$$

or

$$1 = 1$$
$$(\mu_2/\mu_1) = 1$$

Thus, only in the case $\mu_1 = \mu_2$ does the Helmholtz particular solution meet both boundary conditions. If $\mu_1 \neq \mu_2$, we must add to the particular solution some amount of homogeneous solution of $\nabla^2_{2D}\,A_z(r,\theta) = 0$. So we then write generally,

$$A_z(r) = A_z^H(r) + A_z^{homo}(r) \qquad \text{(G.4.14)}$$

where we know that $A_z^{homo}(r)$ can only have terms $\alpha + \beta \ln r$. We then write for the two regions

$A_z(r) = - I\mu_2/(2\pi) \ln(r) + \alpha + \beta \ln r$ r>a

$A_z(r) = - I\mu_2/(\pi a^2) \{ (1/4) (r^2-a^2) + (1/2) a^2 \ln a \} + \alpha' + \beta' \ln r$ r<a

As earlier, we choose the zero point for $A_z(r)$ by requiring that the large r behavior be K ln(r) without a constant added, which then means $\alpha = 0$. And for r<a we must have $\beta' = 0$ to be finite at r = 0. So

$A_z(r) = [- I\mu_2/(2\pi) + \beta] \ln r$ r>a

$A_z(r) = - I\mu_2/(\pi a^2) \{ (1/4) (r^2-a^2) + (1/2) a^2 \ln a \} + \alpha'$ r<a

(G.4.15)

$-\partial_r A_z(r) = [I\mu_2/(2\pi) - \beta](1/r)$ r>a

$-\partial_r A_z(r) = I\mu_2/(\pi a^2) (2r)$ r<a

The boundary conditions are then

$[A_z(a)]^1 = [A_z(a)]^2$

$(1/\mu_1) [(-\partial_r A_z)(a)]^1 = (1/\mu_2) [(-\partial_r A_z)(a)]^2$ // (G.2.6) repeated

or

$[- I\mu_2/(2\pi) + \beta] \ln a = - I\mu_2/(\pi a^2) \{ (1/4) (a^2-a^2) + (1/2) a^2 \ln a \} + \alpha'$

$(1/\mu_1)[I\mu_2/(2\pi) - \beta](1/a) = (1/\mu_2) I\mu_2/(\pi a^2) (2a)$ // insert expressions, set r = a

or

$[- I\mu_2/(2\pi) + \beta] \ln a = - I\mu_2/(2\pi) \ln a + \alpha'$

$(1/\mu_1)[I\mu_2/(2\pi) - \beta](1/a) = I/(2\pi a)$ // simplify

or

$\beta \ln a = \alpha'$

$I\mu_2/(2\pi) - \beta = \mu_1 I/(2\pi)$ // simplify some more

so we find that

$\beta = I/(2\pi) (\mu_2-\mu_1)$

$\alpha' = I/(2\pi) (\mu_2-\mu_1) \ln a$ (G.4.16)

The full solution is then

$A_z(r) = [- I\mu_2/(2\pi) + \beta] \ln r$ r>a

$A_z(r) = - I\mu_2/(\pi a^2) \{ (1/4) (r^2-a^2) + (1/2) a^2 \ln a \} + \alpha'$ r<a

or

$A_z(r) = [- I\mu_2/(2\pi) + \{I/(2\pi) (\mu_2-\mu_1)\}] \ln r$ r>a

$A_z(r) = - I\mu_2/(\pi a^2) \{ (1/4) (r^2-a^2) + (1/2) a^2 \ln a \} + \{ I/(2\pi) (\mu_2-\mu_1) \ln a \}$ r<a

or

$A_z(r) = I/(2\pi) [- \mu_2 + (\mu_2-\mu_1)] \ln r$ r>a

$A_z(r) = - I\mu_2/(\pi a^2) (1/4) (r^2-a^2) - I\mu_2/(\pi a^2) (1/2) a^2 \ln a + I/(2\pi) (\mu_2-\mu_1) \ln a$ r<a

or

$A_z(r) = - (I/(2\pi) [\mu_1] \ln r$ r>a

$A_z(r) = - I\mu_2/(\pi a^2) (1/4) (r^2-a^2) - I\mu_1/(\pi) (1/2) \ln a$ r<a

or

$$A_z(r) = -[I\mu_1/2\pi] \ln r \qquad r>a$$
$$A_z(r) = -[I\mu_2/(4\pi a^2)](r^2-a^2) - [I\mu_1/2\pi] \ln a \qquad r<a \qquad \text{(G.4.17)}$$

This result matches the results (G.2.7) and (G.3.3) of the previous two methods and then results in the B field solution

$$B_\theta = -\partial_r A_z = [I\mu_1/2\pi]\partial_r \ln(r) = [I\mu_1/2\pi](1/r) \qquad r > a \quad \text{region 1}$$
$$B_\theta = -\partial_r A_z = [I\mu_2/(2\pi a^2)]\, r = [I\mu_2/2\pi](r/a^2) \qquad r < a \quad \text{region 2} \qquad \textit{(G.2.8)}$$