This is about reusing Q(r',r) from Appendix B. All is installed.

G.4 Solution for A_z using the 2D Helmholtz Integral

This method of finding A_z is technically more difficult than the first two methods shown in Section G.2 (solving the ODE and adding a homogeneous solution to meet the boundary conditions) and Section G.3 (instant Ampere's Law solution). The method is important because our entire Chapter 4 is based on using Helmholtz integrals to develop the theory of transmission line parameters, and this is one Helmholtz integral that we can actually compute without too much effort. An important result we find is that, when $\mu_1 \neq \mu_2$, the Helmholtz integral by itself does not supply the complete solution, and one must add in some amount of homogeneous solution of $\nabla^2_{2D} A_z(r) = 0$ to meet the required boundary conditions at r=a.

We really have a Poisson integral since $\beta_1^2 = 0$, but the full Helmholtz integral works the same way so we keep referring to it as a Helmholtz integral.

Recall from above:

$$\nabla^2_{2D} A_z(r) = - [I\mu_2/(\pi a^2)]\, \theta(r \leq a) \qquad \mathbf{B} = B_\theta \hat{\boldsymbol{\theta}} \quad \text{with} \quad B_\theta = - \partial_r A_z \,. \qquad (G.1.7)$$

Using the 2D free-space Green's function (propagator) as reviewed in Appendix I equation (I.1.6),

$$g(x|x') = (1/2\pi) \ln(1/R) = - (1/4\pi) \ln(R^2) \qquad R = R = |\mathbf{x}-\mathbf{x}'| \,, \qquad (G.4.1)$$

we may write the particular solution to (G.1.7) as the following "Helmholtz" integral [see (I.1.8)]

$$-A_z^H(r) = \int dA' \, [(1/4\pi) \ln(R^2)] \, [I\mu_2/(\pi a^2)] \, \theta(r \leq a)$$

$$= \int_0^a r' \, dr' \int_{-\pi}^{\pi} d\theta' \, [(1/4\pi) \ln(r^2 + r'^2 - 2rr'\cos(\theta-\theta'))] \, [I\mu_2/(\pi a^2)]$$

$$= (1/4\pi) \, [I\mu_2/(\pi a^2)] \int_0^a r' \, dr' \, [\int_{-\pi}^{\pi} d\theta' \ln(r^2 + r'^2 - 2rr'\cos(\theta-\theta')) \,]$$

$$= \frac{\mu_2 I}{4\pi^2 a^2} \int_0^a r' \, dr' \, [\, 2\, Q(r',r) \,] \qquad (G.4.2)$$

where

$$Q(r',r) \equiv (1/2) \int_{-\pi}^{\pi} d\theta' \ln(r^2 + r'^2 - 2rr'\cos(\theta-\theta')) = (1/2) \int_{-\pi}^{\pi} d\theta'' \ln(r^2 + r'^2 - 2rr'\cos(\theta''))$$

$$= \int_0^{\pi} dx \ln(r^2 + r'^2 - 2rr'\cos x) \,. \qquad (G.4.3)$$

But we have already computed this $A_z^H(r)$ in Appendix B where it was called $A_z^{(c)}(r,\theta)$, see (B.7.3) and (B.7.4). We may therefore borrow the solution (B.7.7) to obtain the results,

$$A_z^H(r>a) = -\frac{\mu_2 I}{\pi a^2}\{(1/2)\, a^2 \ln r\} = -\frac{\mu_2 I}{2\pi}\ln r$$

$$A_z^H(r<a) = \frac{\mu_2 I}{2\pi a^2}\{(a^2-r^2)/2 - a^2 \ln a\}\ . \qquad \text{(G.4.4)}$$

The derivatives are

$$\partial_r A_z^H(r>a) = -\frac{\mu_2 I}{2\pi r}$$

$$\partial_r A_z^H(r<a) = -\frac{\mu_2 I}{2\pi a^2}\, r\ . \qquad \text{(G.4.5)}$$

Recall the two boundary conditions,

$$[A_z(a)]^1 = [A_z(a)]^2\ .$$
$$(1/\mu_1)\,[(\partial_r A_z)(a)]^1 = (1/\mu_2)\,[(\partial_r A_z)(a)]^2\ . \qquad \textit{(G.2.6)}$$

For our particular Helmholtz integral $A_z^H(r)$ we evaluate these boundary conditions to find

$$-\frac{\mu_2 I}{2\pi}\ln a = \frac{\mu_2 I}{2\pi a^2}\{(a^2-a)/2 - a^2 \ln a\}$$

$$(1/\mu_1)\left[-\frac{\mu_2 I}{2\pi a}\right] = (1/\mu_2)\left[-\frac{\mu_2 I}{2\pi a^2}\, a\right]$$

or

$$1 = 1$$
$$(\mu_2/\mu_1) = 1\ . \qquad \text{(G.4.6)}$$

Thus, only in the case $\mu_1 = \mu_2$ does the Helmholtz particular solution meet both boundary conditions. If $\mu_1 \neq \mu_2$, we must add to the particular solution some amount of homogeneous solution of $\nabla^2_{2D} A_z(r,\theta) = 0$. So we then write generally,

$$A_z(r) = A_z^H(r) + A_z^{homo}(r) \qquad \text{(G.4.7)}$$

where we know that $A_z^{homo}(r)$ can only have terms $\alpha + \beta \ln r$. We then write for the two regions

$$A_z(r) = -\frac{\mu_2 I}{2\pi}\ln(r) + \alpha + \beta \ln r \qquad r>a$$

$$A_z(r) = \frac{\mu_2 I}{2\pi a^2}\{(a^2-r^2)/2 - a^2 \ln a\} + \alpha' + \beta' \ln r \qquad r<a\ . \qquad \text{(G.4.8)}$$

As earlier, we choose the zero point for $A_z(r)$ by requiring that the large r behavior be K ln(r) without a constant added, which then means $\alpha = 0$. And for r<a we must have $\beta' = 0$ to be finite at r = 0. So

$$A_z(r) = -\frac{\mu_2 I}{2\pi} \ln(r) + \beta \ln r \qquad r>a$$

$$A_z(r) = \frac{\mu_2 I}{2\pi a^2} \{ (a^2-r^2)/2 - a^2 \ln a \} + \alpha' \qquad r<a \qquad \text{(G.4.9)}$$

$$-\partial_r A_z(r) = [\frac{\mu_2 I}{2\pi} - \beta](1/r) \qquad r>a$$

$$-\partial_r A_z(r) = \frac{\mu_2 I}{\pi a^2} (2r) \qquad r<a \ . \qquad \text{(G.4.10)}$$

The boundary conditions are then

$$[A_z(a)]^1 = [A_z(a)]^2$$
$$(1/\mu_1) [(-\partial_r A_z)(a)]^1 = (1/\mu_2) [(-\partial_r A_z)(a)]^2 \qquad \textit{(G.2.6)}$$

or

$$-\frac{\mu_2 I}{2\pi} \ln(a) + \beta \ln a = \frac{\mu_2 I}{2\pi a^2} \{ (a^2-a^2)/2 - a^2 \ln a \} + \alpha'$$

$$(1/\mu_1) [\frac{\mu_2 I}{2\pi} - \beta](1/a) = (1/\mu_2) \frac{\mu_2 I}{\pi a^2} (2a)$$

or

$$[- I\mu_2/(2\pi) + \beta] \ln a = - I\mu_2/(2\pi) \ln a + \alpha'$$
$$(1/\mu_1)[I\mu_2/(2\pi) - \beta](1/a) = I/(2\pi a) \qquad \text{// simplify}$$

or

$$\beta \ln a = \alpha'$$
$$I\mu_2/(2\pi) - \beta = \mu_1 I/(2\pi) \qquad \text{// simplify some more}$$

so we find that

$$\beta = I/(2\pi) (\mu_2-\mu_1)$$
$$\alpha' = I/(2\pi) (\mu_2-\mu_1) \ln a \ . \qquad \text{(G.4.11)}$$

The full solution is then

$$A_z(r) = [- I\mu_2/(2\pi) + \beta] \ln r \qquad r>a$$
$$A_z(r) = - I\mu_2/(\pi a^2) \{ (1/4) (r^2-a^2) + (1/2) a^2 \ln a \} + \alpha' \qquad r<a$$

or

$$A_z(r) = [- I\mu_2/(2\pi) + \{I/(2\pi) (\mu_2-\mu_1)\}] \ln r \qquad r>a$$
$$A_z(r) = - I\mu_2/(\pi a^2) \{ (1/4) (r^2-a^2) + (1/2) a^2 \ln a \} + \{ I/(2\pi) (\mu_2-\mu_1) \ln a \} \qquad r<a$$

or

$$A_z(r) = I/(2\pi) [- \mu_2 + (\mu_2-\mu_1)] \ln r \qquad r>a$$
$$A_z(r) = - I\mu_2/(\pi a^2) (1/4) (r^2-a^2) - I\mu_2/(\pi a^2) (1/2) a^2 \ln a + I/(2\pi) (\mu_2-\mu_1) \ln a \qquad r<a$$

or

$A_z(r) = -(I/(2\pi)[\mu_1]\ln r$ $\quad r>a$

$A_z(r) = -I\mu_2/(\pi a^2)(1/4)(r^2-a^2) - I\mu_1/(\pi)(1/2)\ln a$ $\quad r<a$

or

$A_z(r) = -[I\mu_1/2\pi]\ln r$ $\quad r>a$

$A_z(r) = -[I\mu_2/(4\pi a^2)](r^2-a^2) - [I\mu_1/2\pi]\ln a$ $\quad r<a$. (G.4.12)

This result matches the results (G.2.7) and (G.3.3) of the previous two methods and then results in the B field solution,

$B_\theta = -\partial_r A_z = [I\mu_1/2\pi]\partial_r \ln(r) = [I\mu_1/2\pi](1/r)$ $\quad r > a$ region 1

$B_\theta = -\partial_r A_z = [I\mu_2/(2\pi a^2)]\, r = [I\mu_2/2\pi](r/a^2)$ $\quad r < a$ region 2 . *(G.2.8)*