**The Mu Paradox** PhL 11.6.13

This nano-detail could cost me a month! We shall see.

I resolved the paradox on 11.7.13, it did not take month, just a day. I am adding comments in red below. // All done. There really is a big paradox if you don't account for surface currents!

**Statement of the Paradox.**

**On the One Hand:** My current derivation of this equation [ surface currents missing ]

$$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2 - \mu_1\sigma_1)\,\mathbf{A} = -\Sigma_{i=2}^{N}\mu_i\mathbf{J}_i \qquad \text{all of region R} \qquad (1.3.23)$$

clearly has the $\mu_i$ associated with the $\mu$ inside conductor i and these $\mu_i \neq \mu_1$ in the dielectric. [ correct] This result occurs when I use the "region-1 King gauge" inside both dielectric and inside all conductors:

$$\text{div}\,\mathbf{A} = -\mu_1\varepsilon_1\,\partial_t\varphi - \mu_1\sigma_1\varphi \qquad \text{// applies to all of R} \qquad (1.3.18)$$

If I then work through the W(z) section of Chapter 4, I end up with the wire having the following external inductance

$$L_e = \frac{\mu}{2\pi}\,\{\int_{C_1} dx'\,dy'\,b_1(x',y')\,\ln(s2/s1) - \int_{C_2} dx'\,dy'\,b_2(x',y')\,\ln(s2/s1)\,\} \qquad (4.4.8)$$

where $\mu$ is that for the *conductor*, not the dielectric. The integral is dimensionless and the $b_i$ are dimensionless transverse currents which integrate to 1 across each conductor. Continuing, I end up finding that $L_e = (\mu/2\pi)K$ where K is a geometric integral and I thus end up with $\mu$ for the *conductor* inside this formula for $L_e$. [ if you omit surface currents, the conclusion is correct].

**On the Other Hand:** In appendix C I compute $L_e$ by computing the stored energy in the dielectric, which of course has $\mu_1$ . The resulting formula is

$$L_e = \frac{\mu_1}{8\pi}\,\ln\frac{R}{a}\,. \qquad (C.3.8)$$

where $\mu_1$ is for the medium in which that energy was stored.

**External verification?** Which is the correct answer? Matick p 97 computes $L_{int}$ and the $\mu$ appearing there is that of the wire. Matick p 315 (8.11) then gives the external inductance where $\mu$ is for the dielectric. I looked on the web a bit, this is surely the right answer. [ yes, c.3.8 with $\mu_1$ = diel is correct.]

So something is wrong with my development [yes] , and I already have an idea or two and it has to do with my choice of gauge [no, the gauge is OK]. Consider these two equations

$$\text{div}\,\mathbf{A} = -\mu_1\varepsilon_1\,\partial_t\varphi - \mu_1\sigma_1\varphi \qquad \text{// applies to all of R} \qquad (1.3.18)$$

$\mathbf{E} = -\text{ grad } \varphi - \partial_t \mathbf{A}$

$\mathbf{B} = \text{curl } \mathbf{A}$.

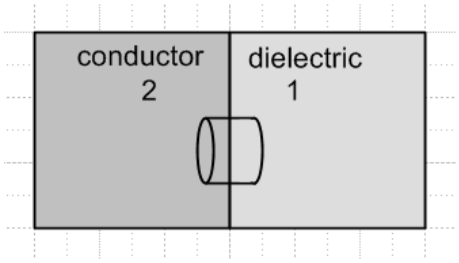
**Plan A**. Proof that **A** in the region-1-King-gauge is completely continuous at a boundary

[ I think this proof is OK, but it does not really bear on the paradox at hand. ]

I want to study things at the boundary between conductor and dielectric. Work in the ω domain and we then have for the first equation

$\text{div } \mathbf{A} = -j\omega\mu_1\xi_1\varphi = -j(\beta_1^2/\omega)\,\varphi$ // King gauge

Now do an integral form for div **A** at the 1/2 boundary:




$\int \text{div } \mathbf{A}\, dV = \int \mathbf{A} \bullet d\mathbf{S}$

$\text{LHS} = \int -j(\beta_1^2/\omega)\,\varphi\, dV = -j(\beta_1^2/\omega) \int \varphi\, dV$

which seems rather strange. If φ is continuous at the boundary which I certainly think is true, then we have $\int \varphi\, dV = sA\,\varphi$ where pillbox has area A and height s. Meanwhile,.

$\text{RHS} = \int \mathbf{A} \bullet d\mathbf{S} = A_{n1} A - A_{n2} A$

and therefore

$-j(\beta_1^2/\omega)\, sA\, \varphi = (A_{n1} - A_{n2})A$

$-j(\beta_1^2/\omega)\, s\, \varphi = (A_{n1} - A_{n2})$ .

Then take $s \to 0$ and conclude that

$A_{n1} = A_{n2}$ [ derivation seems OK, makes use of the gauge condition ]

which says the normal A is continuous at the 1,2 boundary. In the T line app, this A is not of much interest since T line uses a *transverse* A ($A_z$), but at least we have a little result here. This is in my special gauge.

Notice that if the gauge were different in the two regions, then I would get

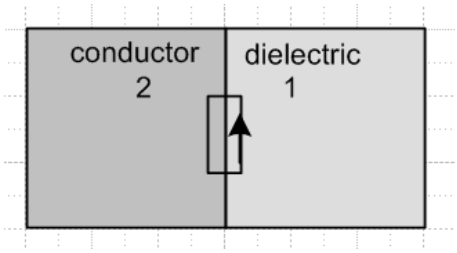
$$\text{LHS} = \int \text{div } \mathbf{A}\, dV = \{ [-j(\beta_1^2/\omega)\, \varphi] - [-j(\beta_2^2/\omega)\, \varphi] \}/2 * V$$

and then $A_n$ would NOT be continuous at the boundary. [ good reason to use one gauge everywhere]

Now look at **B** = curl **A** and see what that has to say:

$$\int_S (\nabla \times \mathbf{A}) \bullet d\mathbf{S} = \oint \mathbf{A} \bullet d\mathbf{s} = \int_S \mathbf{B} \bullet d\mathbf{S}$$

Apply this to the usual little loop at a boundary of width s and height L.




Then (region 1 on right)

$$\oint \mathbf{A} \bullet d\mathbf{s} = (\mathbf{A}_{t1} - A_{t2})L$$

$$\int_S \mathbf{B} \bullet d\mathbf{S} = (B_{t1}+B_{t2})/2 * sL \qquad // F_t \text{ points at viewer}$$

[ Note: $B_{t1}$ is discontinuous at the boundary, but it is no infinite, so argument is OK ]

Then

$$(B_{t1}+B_{t2})/2 * sL = (\mathbf{A}_{t1} - A_{t2})L$$

$$(B_{t1}+B_{t2})/2 * s = (\mathbf{A}_{t1} - A_{t2})$$

Then as s→0 we seem to get

$$A_{t1} = A_{t2}$$ [ derivation seems OK, gauge not used for this part ]

I seem to conclude that all components of **A** are continuous at such a boundary. [ yes ]
It is true that the $B_t$ field jumps,

$$(1/\mu_1)B_{1t} = (1/\mu_2)B_{2t}$$

but that does not affect the above discussion.

**Fact**: In the region-1-King gauge, **A** is continuous at a dielectric/conductor boundary. [ yes ]

I regard that as a good thing really, since it is like φ. Now consider

$$\mathbf{E} = -\text{ grad } \varphi - j\omega\mathbf{A}$$

If **A** is continuous at a boundary, then this is consistent with **E** taking a jump because φ changes slope.

**Corollary**: When I compute W(z) as $\Delta A_z$ in Chapter 4, there will be no issue with $A_z$ taking a sudden jump at the conductor surfaces, so that is not going to "explain" my paradox. [ correct ]

**Plan B**. Maybe setting **E** = 0 inside the conductor the way I did it is wrong. [ no, this was not it ]

I have this

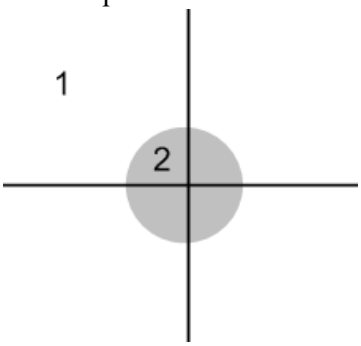
$$(\nabla^2 - \mu_2\varepsilon_2\,\partial_t^2)\,\mathbf{A} = [\,(\mu_2\varepsilon_2 - \mu_1\varepsilon_1)\,\partial_t - \mu_1\sigma_1)\,]\,(-\mathbf{E}-\partial_t\mathbf{A}) - \mu_2\mathbf{J} \qquad \text{// using (1.3.1)}$$

Now what happens if I set $\mathbf{E} = \mathbf{J}/\sigma_2$ which is exactly correct rather than **E** = 0. This will create following *new term* on the right of my final equation

$$[\,(\mu_2\varepsilon_2 - \mu_1\varepsilon_1)\,\partial_t - \mu_1\sigma_1)\,]\,(-\mathbf{J}/\sigma_2)$$

which is a big mess, but then I would say $\sigma_2 \to \infty$ and this entire term vanishes and I am back.

**Plan C.** OK, let's try to do the *single round wire all by itself* using my two equations and see what happens. [ in other words, try the case n = 2 ] Here is our picture of the wire cross section,




And here are the two equations in my special gauge: [ surface current missing for region 1 ]

$$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2 - \mu_1\sigma_1)\,\mathbf{A} = 0 \qquad \text{region 1}$$

$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1\partial_t\ )\ \mathbf{A} = -\mu_2\mathbf{J}_2$ region 2 time domain

$(\nabla^2+\beta_1^2)\ \mathbf{A} = 0$ region 1

$(\nabla^2+\beta_1^2)\ \mathbf{A} = -\mu_2\mathbf{J}_2$ region 2 ω domain

I would then claim a common equation for both regions

$(\nabla^2+\beta_1^2)\ \mathbf{A} = -\mu_2\mathbf{J}_2$ region 1 union region 2

Now I claim somehow that we can neglect $A_t$ since $J_t \approx 0$ and we then have

$(\nabla^2+\beta_1^2)\ A_z = -\mu_2 J_{z2}$ region 1 union region 2

and we then get to treat this as a scalar wave equation like φ [correct] and we get

$$A_z(\mathbf{x},\omega) = +\frac{\mu_2}{4\pi}\int J_{z2}(\mathbf{x}',\omega)\frac{\exp(-j\beta_1 R)}{R}\,dV' \qquad R = |\mathbf{x} - \mathbf{x}'| \quad . \qquad (1.5.9)$$

where I suppose **x** can be *anywhere*, but the integral is only over region 2. At low ω this should be

$$A_z(\mathbf{x},\omega) = +\frac{\mu_2}{4\pi}\int J_{z2}(\mathbf{x}',\omega)\frac{1}{R}\,dx'dy'dz' \qquad R = |\mathbf{x} - \mathbf{x}'|$$

I can now jump to Appendix C.4 and result this discussion. There I deal with the divergence issues of this integral. I get this temp result

$$A_z(r) = -\frac{\mu J_z}{2\pi}\int_0^a r'\,dr'\int_0^{2\pi} d\theta\,\ln\left(\sqrt{r'^2+r^2-2rr'\cos\theta}\,/\Lambda\right) \qquad (C.4.4)$$

This integral I actually do in my "verifications doc" with the solution reviewed as a "reader exercise". Now this is very interesting. In that document I start with

$$A_z(\mathbf{x},\omega) = \frac{\mu}{4\pi}\int J_{z,a}(\mathbf{x}',\omega)\frac{e^{-j\beta(\omega)R}}{R}dV' \qquad R = |\mathbf{x} - \mathbf{x}'| \qquad (1.5.9)_z$$

which has a "certain μ" that you can plainly see. I then boil this down to

$$A_z(r) = -\frac{\mu J_z}{2\pi}\int_0^a r'\,dr'\int_0^{2\pi} d\theta\,\ln\left(\sqrt{r'^2+r^2-2rr'\cos\theta}\,/\Lambda\right) \qquad (C.4.4)$$

I then show that

$$\int_0^{2\pi} d\theta \ln\left(\sqrt{r'^2+r^2-2rr'\cos\theta}\ \right) = \pi \ln\left[\frac{(r'^2+r^2)+|r'^2-r^2|}{2}\right].$$

where the important abs value is visible. I obtain these results

$A_z(r) = -(\mu_1 J_z/2)\ a^2 \ln(r)$ $\quad r > a$ // outside wire (*)
$B(r) = +(\mu_1 J_z a^2/2)/r$
$H(r) = B(r)/\mu_1 = (J_z a^2/2)/r = (I/\pi a^2)\ a^2/(2r) = (I/2\pi)\ r^{-1}$

$A_z(r) = -(\mu_2 J_z/4)\ r^2$ $\quad r < a$ // inside wire
$B(r) = (\mu_2 J_z/2)\ r$
$H(r) = B(r)/\mu_2 = (J_z/2)\ r = (I/2\pi a^2)\ r$

I have assigned the μ's above *as they must be* in order to agree with results below, BUT, it seems that you have to have $\mu_1 = \mu_2 = \mu$ since $A_z(r)$ came from a single initial equation!! So how can the μ's be the same and be different at the same time? [ this is the paradox ]

In that same appendix I do this for internal inductance (2 = conductor)

$dU = (1/2)\ \mu_2 H^2\ d\mathcal{V}$ . (C.3.1)

$U_i = (1/2)\ (L_i\ dz)\ I^2$ . (C.3.2)

$U_i = \int_{inside} (1/2)\ \mu_2 H^2 d\mathcal{V}$,

$H(r) = (I/2\pi a^2)\ r \qquad r \le a$ . (C.3.3)

$U_i = (\mu_2 dz\ I^2/16\pi) = (1/2)\ [\mu_2/8\pi]\ dz\ I^2$

$L_i = \mu_2/8\pi$

So there is $\mu_2$ properly sitting in the internal L formula, all seems clean here.

Now here is my external L calculation:

$H(r) = (I/2\pi)\ r^{-1}$ . (C.3.6)

$dU(r) = (1/2)\ \mu_1 H^2 d\mathcal{V}$

$L_e = \frac{\mu_1}{8\pi} \ln\frac{R}{a}$

Just for the record here,

$H_{1t} = H_{2t}$ or $(1/\mu_1)B_{1t} = (1/\mu_2)B_{2t}$ *(1.1.26)*

$B_{n1} = B_{n2}$ or $\mu_1 H_{n1} = \mu_2 H_{n2}$ *(1.1.29)*

So here is the Big Mystery: To get $L_i$ and $L_e$ to come out right, you have to have $\mu_1$ and $\mu_2$ separately. But my unified $A_z(r)$ from which both results derive can only have a single $\mu$ sitting in front! [ yes, it is a mystery of you leave off the surface current ]

**Idea**: Maybe there is a surface magnetization current [ ding! ]that I have ignored, analogous to the polarization charge at the boundary between two dielectric media?

Since I keep saying low $\omega$, why not just set $\omega = 0$ and $\beta_1 = 0$ and then it is just a Poisson type problem. Yes, let's just do the magnetostatics problem [ that was the correct decision ] and see what falls out. I will power this up in a separate doc.