Appendix H : Calculation of $\nabla^2[1/r]$, $\nabla^2[h(r)/r]$ and $\nabla^2[e^{-jkr}/r]$

This has been installed, no need to keep long term.

Theorem 1: $\nabla^2[1/r] = -4\pi\delta(\mathbf{r})$ (H.1)

Proof: Let V be some volume that contains the origin point $\mathbf{r} = 0$. Carve out from V a small spherical cavity of radius a surrounding $\mathbf{r} = 0$. If we call this spherical volume V_a and then $V' = V - V_a$ is the original volume with the spherical cavity. In order to show that some function $g(r) = \delta(\mathbf{r})$, one has to show that

$$\lim_{a\to 0} \int_{V'} dV\, g(r) = 0 \qquad \text{(H.2a)}$$

$$\lim_{a\to 0} \int_{V_a} dV\, g(r) = 1 \qquad \text{(H.2b)}$$

This is basically the definition of $\delta(\mathbf{r})$.

Our function of interest is

$$g(r) = -(1/4\pi)\,\nabla^2[1/r] \qquad \text{(H.3)}$$

Using ∇^2 in spherical coordinates acting on a function of r, one finds that

$$\nabla^2[1/r] = (1/r^2)\partial_r(r^2\partial_r)\,[1/r] = 0 \qquad r > 0 \qquad \text{(H.4)}$$

so that

$$g(r) = -(1/4\pi)\,\nabla^2[1/r] = 0 \qquad r > 0 \qquad \text{(H.5)}$$

Thus, (H.2a) is trivially satisfied since $r > 0$ everywhere in volume V'.

It remains to show (H.2b). Consider the integral appearing in the left side of (H.2b)

$$\int_{V_a} dV\, g(r) = -(1/4\pi) \int_{V_a} dV\, \nabla^2[1/r] = -(1/4\pi) \int_{V_a} dV\, \nabla \bullet \nabla[1/r] \quad . \qquad \text{(H.6)}$$

The divergence theorem says

$$\int_V dV\, \text{div}\, \mathbf{F} = \int_S d\mathbf{S} \bullet \mathbf{F} \qquad \text{(H.7)}$$

where V is any closed volume whose surface is S, and d**S** points out. Using $V = V_a$ and $\mathbf{F} = \nabla[1/r]$,

$$\int_{Va} dV\, g(r) = -(1/4\pi) \int_{Sa} d\mathbf{S} \bullet \nabla[1/r] = -(1/4\pi)\ [4\pi a^2\, \hat{\mathbf{r}}] \bullet \nabla[1/r]$$

$$= -a^2\, \hat{\mathbf{r}} \bullet [-r^{-2}]^{r=a}\, \hat{\mathbf{r}} = a^2\, a^{-2} = 1\ .$$

Thus,

$$\lim_{a\to 0} \int_{Va} dV\, g(r) = 1$$

and we have then verified (H.2b). Therefore we conclude that

$$-(1/4\pi)\nabla^2[1/r] = \delta(\mathbf{r}) \qquad \text{(H.8)}$$

or

$$\nabla^2[1/r] = -4\pi\delta(\mathbf{r})$$

which is (H.1). **QED**

Theorem 2: $\nabla^2[h(r)/r] = -4\pi\, h(0)\, \delta(\mathbf{r}) + h''(r)/r$ (H.9)

Proof: Start with this vector identity

$$\nabla^2(\varphi\psi) = \varphi\nabla^2\psi + \psi\nabla^2\varphi + 2\, \nabla\varphi \bullet \nabla\psi \qquad \text{(H.10)}$$

so that

$$\nabla^2(h\, r^{-1}) = h\nabla^2(r^{-1}) + r^{-1}\nabla^2 h + 2\ \nabla h \bullet \nabla(r^{-1})$$

$$= -h\, 4\pi\, \delta(\mathbf{r}) + r^{-1}\nabla^2 h + 2\, [\, h'\, \hat{\mathbf{r}} \bullet (-r^{-2})\, \hat{\mathbf{r}}$$

$$= -4\pi\, h(0)\, \delta(\mathbf{r}) + r^{-1}\nabla^2 h - 2\, r^{-2}\ h'(r) \qquad \text{(H.11)}$$

Algebra shows that,

$$\nabla^2 h = (1/r^2)\partial_r(r^2\partial_r)h(r) = h''(r) + (2/r)h'(r) \qquad \text{(H.12)}$$

so then

$$\nabla^2(h\, r^{-1}) = -4\pi\, h(0)\, \delta(\mathbf{r}) + r^{-1}\, [h''(r) + (2/r)h'(r)\,] - 2\, r^{-2}\ h'(r)$$

$= - 4\pi\, h(0)\, \delta(\mathbf{r}) + h''(r)/r$

which is the claim of (H.9). **QED**

Example of Theorem 2: $h(r) = e^{-jkr}$

$h = e^{-jkr}$ $\quad h(0) = 1$ $\quad h' = -jk\, e^{-jkr}$ $\quad h'' = -k^2\, e^{-jkr}$

$\nabla^2(e^{-jkr}/r) = - 4\pi\, 1\, \delta(\mathbf{r}) + [-k^2\, e^{-jkr}\,] / r$

$= -4\pi\delta(r) - k^2(e^{-jkr}/r)$

Thus,

$$(\nabla^2+k^2) (e^{-jkr}/r) = - 4\pi\delta(\mathbf{r}) \qquad \text{(H.13)}$$

or

$$- (\nabla^2+k^2) (e^{-jkr}/4\pi r) = \delta(\mathbf{r}) \qquad \text{(H.14)}$$

If we translate the origin from **0** to **x'** and define $R = |\mathbf{r} - \mathbf{x}'|$ and rename **r** to be **x**, (H.13) becomes

$$- (\nabla^2+k^2) (e^{-jkR}/R) = 4\pi\, \delta(\mathbf{x}-\mathbf{x}') \qquad \text{(H.15)}$$

Thus we may identify e^{-jkR}/R as the Helmholtz equation Green's Function.