11.14.13

Draft of App H. I split old App H into two appendices, one for 2D and one for 3D.

Appendix H : The Poisson and Helmholtz Free-Space Propagators in 3D

Note: Appendix I deals with these propagators in 2D rather than 3D.

H.1 Overview and Meaning of Free-Space Propagators

This appendix proves the following Facts:

Fact 1 : $-\nabla^2[1/4\pi r] = \delta(\mathbf{r})$ *(H.2.1)* (H.1.1)

Fact 2 : $-\nabla^2[h(r)/r] = 4\pi\, h(0)\, \delta(\mathbf{r}) - h''(r)/r$ *(H.3.1)* (H.1.2)

Fact 3 : $-(\nabla^2+k^2)(e^{-jkr}/4\pi r) = \delta(\mathbf{r})$ *(H.3.5)* (H.1.3)

Throughout, ∇^2 is the usual 3D Laplacian operator $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$. In the first and last results above, if one replaces $\mathbf{r} \rightarrow \mathbf{r}-\mathbf{r}'$ (a simple translational shift of origin) ones finds

$-\nabla^2[1/4\pi R] = \delta(\mathbf{r}-\mathbf{r}')$ $\quad R = |\,\mathbf{r}-\mathbf{r}'\,|$ (H.1.4)

$-(\nabla^2+k^2)[e^{-jkR}/4\pi R] = \delta(\mathbf{r}-\mathbf{r}')$ $\quad \delta(\mathbf{r}-\mathbf{r}') = \delta(x-x')\,\delta(y-y')\,\delta(z-z')$ (H.1.5)

The quantities in brackets are known as free-space Green's Functions (Green Functions) or propagators, or as "fundamental solutions":

$1/4\pi R$ = the Poisson 3D free-space propagator (H.1.6)

$e^{-jkR}/4\pi R$ = the Helmholtz 3D free-space propagator (H.1.7)

The significance of these propagators is the following:

$-\nabla^2 f(\mathbf{x}) = s(\mathbf{x})$ $\Rightarrow$ $f(\mathbf{x}) = \int d^3x'\,[1/4\pi R]\, s(\mathbf{x}')$ + homogeneous solutions

The Poisson Equation (H.1.8)

$-(\nabla^2+k^2) f(\mathbf{x}) = s(\mathbf{x})$ $\Rightarrow$ $f(\mathbf{x}) = \int d^3x'\,[e^{-jkR}/4\pi R]\, s(\mathbf{x}')$ + homogeneous solutions

The Helmholtz Equation (H.1.9)

The equations on the left are inhomogeneous partial differential equations driven by source function $s(\mathbf{x})$. If one is careful to include in $s(\mathbf{x})$ all source contributions (such as those on boundary surfaces), one generally does not have to add any homogeneous solutions on the right. A homogeneous solution refers to $-\nabla^2 f_h(\mathbf{x}) = 0$, for example. The solutions shown on the right above can be instantly verified as follows:

$$f(\mathbf{x}) = \int d^3x' \, [1/4\pi R] \, s(\mathbf{x}') + f_h(\mathbf{x})$$

$$-\nabla^2 f(\mathbf{x}) = \int d^3x' \, (-\nabla^2 \, [1/4\pi R] \,) \, s(\mathbf{x}') - \nabla^2 f_h(\mathbf{x}) = \int d^3x' \, \delta(\mathbf{r}-\mathbf{r}') \, s(\mathbf{x}') - 0 = s(\mathbf{x}) \qquad \text{(H.1.10)}$$

and similarly for $-(\nabla^2+k^2) f = g$.

A "free space" Green's Function g_F in general is a solution of

$$D \, g_F(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'), \qquad g_F(\mathbf{r}, \mathbf{r}') \to 0 \text{ as } \mathbf{r} \to \infty \qquad \text{(H.1.11)}$$

where D is some differential operator. The condition on the right says g_F must vanish on the Great Sphere. More generally one can write

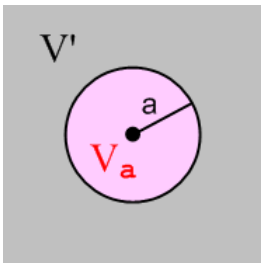
$$D \, g(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'), \qquad g(\mathbf{r}, \mathbf{r}') = 0 \text{ for } \mathbf{r} \text{ on some closed surface enclosing a region of interest} \qquad \text{(H.1.12)}$$

In this second form, the $\int d^3x'$ is over the volume inside that closed surface. We shall not make use of this more general form in this document. George Green (1793-1841), by the way, was an English grain miller.

Looking at $f(\mathbf{x}) = \int d^3x' \, [1/4\pi R] \, s(\mathbf{x}') = \int g_F(\mathbf{x},\mathbf{x}') \, [s(\mathbf{x}') \, d^3x']$, one can say that the kernel Green's Function $g_F(\mathbf{x},\mathbf{x}')$ "propagates" a tiny piece of "source" $[s(\mathbf{x}')d^3x']$ from location $\mathbf{x}'$ to location $\mathbf{x}$ so that the solution $f(\mathbf{x})$ is then a sum of all such propagated contributions as the source ranges over the entire volume of interest, which for us is all 3D space where the source is non-vanishing.

H.2 Fact 1: $\quad -\nabla^2[1/r] = 4\pi\delta(\mathbf{r}) \qquad \text{(H.2.1)}$

Proof: Let volume V be all of 3D space. Carve out from V a small spherical cavity of radius a centered at $\mathbf{r} = 0$. If we call this spherical volume V_a and then $V' = V - V_a$ is the original volume with the spherical cavity carved out:



In order to show that some function $g(\mathbf{r}) = \delta(\mathbf{r})$, one has to show that

$$\lim_{a\to 0} \int_{V'} dV\, g(\mathbf{r}) = 0 \qquad \text{(H.2.2a)}$$

$$\lim_{a\to 0} \int_{V_a} dV\, g(\mathbf{r}) = 1 \qquad \text{(H.2.2b)}$$

This is basically the definition of $\delta(\mathbf{r})$. Since $\delta(\mathbf{r})$ has units L^{-3}, g(r) has units L^{-3}.

Our candidate function of interest is

$$g(r) = -(1/4\pi)\,\nabla^2[1/r]\,. \qquad \text{(H.2.3)}$$

Using ∇^2 in spherical coordinates acting on a function of r, one finds that, since $\partial_r(1) = 0$,

$$\nabla^2[1/r] = (1/r^2)\partial_r(r^2\partial_r)\,[1/r] = 0 \qquad r > 0 \qquad \text{(H.2.4)}$$

so that

$$g(r) = -(1/4\pi)\,\nabla^2[1/r] = 0 \qquad r > 0\,. \qquad \text{(H.2.5)}$$

Thus, condition (H.2.2a) is trivially satisfied since $r > 0$ everywhere in volume V'.

It remains to verify condition (H.2.2b). Consider the integral appearing in the left side of (H.2.2b)

$$\int_{V_a} dV\, g(r) = -(1/4\pi) \int_{V_a} dV\, \nabla^2[1/r] = -(1/4\pi) \int_{V_a} dV\, \boldsymbol{\nabla}\bullet\boldsymbol{\nabla}[1/r]\,. \qquad \text{(H.2.6)}$$

The divergence theorem says

$$\int_V dV\, \mathrm{div}\,\mathbf{F} = \int_S d\mathbf{S}\bullet\mathbf{F} \qquad \text{(H.2.7)}$$

where V is any closed volume whose surface is S, and d**S** points out. Using

$$V = V_a \qquad \text{and} \qquad \mathbf{F} = \boldsymbol{\nabla}[1/r] = \hat{\mathbf{r}}\,\partial_r(1/r) = -r^{-2}\hat{\mathbf{r}}$$

we find that

$$\text{LHS (H.2.7)} = \int_{V_a} dV\, \mathrm{div}\,\boldsymbol{\nabla}[1/r] = \int_{V_a} dV\, \nabla^2[1/r] = \int_{V_a} dV\, [-4\pi g(r)] = -4\pi \int_{V_a} dV\, g(r)$$

$$\text{RHS (H.2.7)} = \int_S d\mathbf{S}\bullet\boldsymbol{\nabla}[1/r] = \int d\Omega\, [a^2\,\hat{\mathbf{r}}]\bullet\boldsymbol{\nabla}[1/r]|^{r=a} = \int d\Omega [a^2\,\hat{\mathbf{r}}]\bullet[-a^{-2}\hat{\mathbf{r}}] = -4\pi$$

which tells us that $\int_{V_a}$ dV g(r) = 1 for any a. Thus,

$$\lim_{a\to 0} \int_{V_a} dV\, g(r) = 1$$

and we have then verified (H.2.2b). Therefore we conclude that the candidate g(**r**) of (H.2.3) is in fact the same as δ(**r**) so

$$-(1/4\pi)\nabla^2[1/r] = \delta(\mathbf{r}) \qquad \text{(H.2.8)}$$

or

$$\nabla^2[1/r] = -4\pi\delta(\mathbf{r}) \qquad \text{(H.2.9)}$$

which is (H.2.1). **QED**

H.3 Fact 2: $\nabla^2[h(r)/r] = -4\pi\, h(0)\, \delta(\mathbf{r}) + h''(r)/r$ (H.3.1)

Proof: Start with this vector identity,

$$\nabla^2(\varphi\psi) = \varphi\nabla^2\psi + \psi\nabla^2\varphi + 2\,\nabla\varphi \bullet \nabla\psi \ . \qquad \text{(H.3.2)}$$

This identity is valid in any number of dimensions (implied sum on i from 1 to N) ,

$$\partial_i^2(\varphi\psi) = \partial_i[\,(\partial_i\varphi)\psi + \psi(\partial_i\varphi)] = (\partial_i^2\varphi)\psi + (\partial_i\varphi)\,(\partial_i\psi) + \varphi(\partial_i^2\psi) + (\partial_i\varphi)\,(\partial_i\psi) \ .$$

So apply (H.3.2) to the case φ = h and ψ = r^{-1},

$$\nabla^2(h\, r^{-1}) = h\nabla^2(r^{-1}) + r^{-1}\nabla^2 h + 2\ \nabla h \bullet \nabla(r^{-1})$$

$$= -h\, 4\pi\, \delta(\mathbf{r}) + r^{-1}\nabla^2 h + 2\,[\, h'\, \hat{\mathbf{r}} \bullet (-r^{-2})\, \hat{\mathbf{r}}\,] \qquad \text{// using (H.2.9)}$$

$$= -4\pi\, h(0)\, \delta(\mathbf{r}) + r^{-1}\nabla^2 h - 2\, r^{-2}\ h'(r) \ . \qquad \text{(H.3.3)}$$

Algebra shows that, using spherical coordinates,

$$\nabla^2 h = (1/r^2)\partial_r(r^2\partial_r)h(r) = h''(r) + (2/r)h'(r) \qquad \text{(H.3.4)}$$

so then

$$\nabla^2(h\, r^{-1}) = -4\pi\, h(0)\, \delta(\mathbf{r}) + r^{-1}\,[h''(r) + (2/r)h'(r)\,] - 2\, r^{-2}\ h'(r)$$

$$= -4\pi\, h(0)\, \delta(\mathbf{r}) + h''(r)/r$$

which is the claim of (H.3.1). **QED**

Fact 3: $- (\nabla^2+k^2) (e^{-jkr}/4\pi r) = \delta(\mathbf{r})$ (H.3.5)

This Fact is just an application of Fact 2 to the case $h(r) = e^{-jkr}$:

$h = e^{-jkr}$ $h(0) = 1$ $h' = -jk\, e^{-jkr}$ $h'' = -k^2\, e^{-jkr}$

$\nabla^2[h(r)/r] = - 4\pi\, h(0)\, \delta(\mathbf{r}) + h''(r)/ r$ *(H.3.1)*

so

$\nabla^2(e^{-jkr}/r) = - 4\pi\, 1\, \delta(\mathbf{r}) + [-k^2\, e^{-jkr}\,] / r$

$= -4\pi\delta(r) - k^2(e^{-jkr}/r)$

Thus,

$(\nabla^2+k^2) (e^{-jkr}/r) = - 4\pi\delta(\mathbf{r})$

or

$- (\nabla^2+k^2) (e^{-jkr}/4\pi r) = \delta(\mathbf{r})$

as claimed.