Save old App H last section which is now in App I

This is just archive, it is in Appendix I now, so throw this out some day.

Theorem 3: $\nabla^2[\ln(1/r)] = -2\pi\delta(\mathbf{r})$ in two dimensions (H.1)'

Our 2D proof here will be an "copy, paste and edit" version of the 3D proof of Theorem 1. We shall use the word "volume" with symbol V to denote *area*. A "spherical cavity" now means a circular hole. It should be understood that $\nabla^2 = \partial_x^2+\partial_y^2$ and $\delta(\mathbf{r})= \delta(x)\delta(y)$. We indicate all edits in red and equation numbers are the same as for Theorem 1 with a prime added.

Proof: Let volume V be all of 2D space. Carve out from V a small spherical cavity of radius a centered at $\mathbf{r} = 0$. If we call this spherical volume V_a and then $V' = V - V_a$ is the original volume with the spherical cavity. In order to show that some function $g(r) = \delta(\mathbf{r})$, one has to show that

$$\lim_{a\to 0} \int_{V'} dV\, g(r) = 0 \qquad \text{(H.2a)'}$$

$$\lim_{a\to 0} \int_{V_a} dV\, g(r) = 1 \qquad \text{(H.2b)'}$$

This is basically the definition of $\delta(\mathbf{r})$. Since $\delta(\mathbf{r})$ has units L^{-2}, g(r) has units L^{-2}.

Comment: When any differential operator like ∇ or ∇^2 is applied to $\ln(r_0/r)$, the result is independent of r_0 so we can always take $r_0 = 1$. For example, $\partial_x [\ln(r_0/r)] = \partial_x [\ln r_0 + \ln(1/r)] = \partial_x \ln(1/r)$. In what follows, ln(r) and ln(1/r) are always acted upon by differential operators, so we can interpret these objects as dimensionless quantities $\ln(r/r_0)$ and $\ln(r_0/r)$. Then it is clear below that dim [g(r)] $= L^{-2}$.

Our candidate function of interest is

$$g(r) = -(1/2\pi)\,\nabla^2[\ln(1/r)] = +(1/2\pi)\,\nabla^2[\ln(r)] \qquad \text{(H.3)'}$$

Using ∇^2 in polar coordinates acting on a function of r, one finds that, since $\partial_r(1) = 0$,

$$\nabla^2[\ln(r)] = (1/r)\partial_r(r\partial_r)[\ln(r)] = 0 \qquad r > 0 \qquad \text{(H.4)'}$$

so that

$$g(r) = -(1/2\pi)\,\nabla^2[\ln(1/r)] = 0 \qquad r > 0 \quad . \qquad \text{(H.5)'}$$

Thus, condition (H.2a)' is trivially satisfied since $r > 0$ everywhere in volume V'.

It remains to verify condition (H.2b)'. Consider the integral appearing in the left side of (H.2b)'

$$\int_{V_a} dV\, g(r) = -(1/2\pi) \int_{V_a} dV\, \nabla^2[\ln(1/r)] = -(1/2\pi) \int_{V_a} dV\, \nabla \bullet \nabla[\ln(1/r)] \quad . \qquad \text{(H.6)'}$$

The divergence theorem says

$$\int_V dV\, \text{div}\, \mathbf{F} = \int_S d\mathbf{S} \bullet \mathbf{F} \qquad \textbf{(H.7)'}$$

where V is any closed volume whose surface is S, and d**S** points out. Using

$$V = V_a \quad \text{and} \quad \mathbf{F} = \nabla[\ln(1/r)] = \hat{\mathbf{r}}\partial_r(\ln(1/r)) = -\hat{\mathbf{r}}\partial_r(\ln r) = [-r^{-1}]\,\hat{\mathbf{r}}$$

we find that [here $d\mathbf{S} = dS\,\hat{r}$ where $dS = a d\theta$ = a piece of circumference of the circle bounding V_a]

$$\text{LHS (H.7)} = \int_{V_a} dV\, \text{div}\, \nabla[\ln(1/r)] = \int_{V_a} dV\, \nabla^2[\ln(1/r)] = \int_{V_a} dV\, [-2\pi g(r)] = -2\pi \int_{V_a} dV\, g(r)$$

$$\text{RHS (H.7)} = \int_S d\mathbf{S} \bullet \nabla[\ln(1/r)] = \int d\theta\, [a\,\hat{\mathbf{r}}] \bullet \nabla[\ln(1/r)]|^{r=a} = \int d\theta\, [a\,\hat{\mathbf{r}}] \bullet [-a^{-1}\hat{\mathbf{r}}] = -2\pi$$

which tells us that $\int_{V_a} dV\, g(r) = 1$ for any a. Thus,

$$\lim_{a\to 0} \int_{V_a} dV\, g(r) = 1$$

and we have then verified (H.2b)'. Therefore we conclude that

$$-(1/2\pi)\nabla^2[\ln(1/r)] = \delta(\mathbf{r}) \qquad \text{(H.8)'}$$

or

$$\nabla^2[\ln(1/r)] = -2\pi\delta(\mathbf{r})$$

which is (H.1)'. **QED**