Saved old Appendix H PhL 11.14.13

I replaced this old App H with a pair of new appendices, App H and App I. They subsume all the results developed in this old App H. So this is just for archive, can be tossed eventually.

Appendix H : Calculation of $\nabla^2[1/r]$, $\nabla^2[h(r)/r]$ and $\nabla^2[e^{-jkr}/r]$, and $\nabla^2_{2D}[\ln(1/r)]$

Theorem 1: $-\nabla^2[1/r] = 4\pi\delta(\mathbf{r})$ (H.1)

Proof: Let volume V be all of 3D space. Carve out from V a small spherical cavity of radius a centered at $\mathbf{r} = 0$. If we call this spherical volume V_a and then $V' = V - V_a$ is the original volume with the spherical cavity. In order to show that some function $g(r) = \delta(\mathbf{r})$, one has to show that

$$\lim_{a\to 0} \int_{V'} dV\, g(r) = 0 \qquad \text{(H.2a)}$$

$$\lim_{a\to 0} \int_{Va} dV\, g(r) = 1 \qquad \text{(H.2b)}$$

This is basically the definition of $\delta(\mathbf{r})$. Since $\delta(\mathbf{r})$ has units L^{-3}, $g(r)$ has units L^{-3}.

Our candidate function of interest is

$$g(r) = -(1/4\pi)\,\nabla^2[1/r] \qquad \text{(H.3)}$$

Using ∇^2 in spherical coordinates acting on a function of r, one finds that, since $\partial_r(1) = 0$,

$$\nabla^2[1/r] = (1/r^2)\partial_r(r^2\partial_r)\,[1/r] = 0 \qquad r > 0 \qquad \text{(H.4)}$$

so that

$$g(r) = -(1/4\pi)\,\nabla^2[1/r] = 0 \qquad r > 0 \quad . \qquad \text{(H.5)}$$

Thus, condition (H.2a) is trivially satisfied since $r > 0$ everywhere in volume V'.

It remains to verify condition (H.2b). Consider the integral appearing in the left side of (H.2b)

$$\int_{Va} dV\, g(r) = -(1/4\pi) \int_{Va} dV\, \nabla^2[1/r] = -(1/4\pi) \int_{Va} dV\, \nabla \bullet \nabla[1/r] \quad . \qquad \text{(H.6)}$$

The divergence theorem says

$$\int_V dV \, \text{div}\, \mathbf{F} = \int_S d\mathbf{S} \bullet \mathbf{F} \qquad \textbf{(H.7)}$$

where V is any closed volume whose surface is S, and d**S** points out. Using

$$V = V_a \qquad \text{and} \qquad \mathbf{F} = \nabla[1/r] = \hat{\mathbf{r}}\partial_r(1/r) = -r^{-2}\hat{\mathbf{r}}$$

we find that

$$\text{LHS (H.7)} = \int_{V_a} dV \, \text{div}\, \nabla[1/r] = \int_{V_a} dV \, \nabla^2[1/r] = \int_{V_a} dV \, [-4\pi g(r)] = -4\pi \int_{V_a} dV \, g(r)$$

$$\text{RHS (H.7)} = \int_S d\mathbf{S} \bullet \nabla[1/r] = \int d\Omega \, [a^2 \, \hat{\mathbf{r}}] \bullet \nabla[1/r]|^{r=a} = \int d\Omega [a^2 \, \hat{\mathbf{r}}] \bullet [-a^{-2}\hat{\mathbf{r}}] = -4\pi$$

which tells us that $\int_{V_a} dV \, g(r) = 1$ for any a. Thus,

$$\lim_{a\to 0} \int_{V_a} dV \, g(r) = 1$$

and we have then verified (H.2b). Therefore we conclude that

$$-(1/4\pi)\nabla^2[1/r] = \delta(\mathbf{r}) \qquad \text{(H.8)}$$

or

$$\nabla^2[1/r] = -4\pi\delta(\mathbf{r})$$

which is (H.1). **QED**

Theorem 2: $\nabla^2[h(r)/r] = -4\pi \, h(0) \, \delta(\mathbf{r}) + h''(r)/r$ (H.9)

Proof: Start with this vector identity

$$\nabla^2(\varphi\psi) = \varphi\nabla^2\psi + \psi\nabla^2\varphi + 2\,\nabla\varphi \bullet \nabla\psi \qquad \text{(H.10)}$$

so that

$$\nabla^2(h\, r^{-1}) = h\nabla^2(r^{-1}) + r^{-1}\nabla^2 h + 2\,\nabla h \bullet \nabla(r^{-1})$$

$$= -h\, 4\pi\, \delta(\mathbf{r}) + r^{-1}\nabla^2 h + 2\,[\, h'\, \hat{\mathbf{r}} \bullet (-r^{-2})\, \hat{\mathbf{r}}$$

$$= -4\pi \, h(0) \, \delta(\mathbf{r}) + r^{-1}\nabla^2 h - 2\, r^{-2}\, h'(r) \qquad \text{(H.11)}$$

Algebra shows that,

$$\nabla^2 h = (1/r^2)\partial_{\mathbf{r}}(r^2\partial_{\mathbf{r}})h(r) = h''(r) + (2/r)h'(r) \qquad \text{(H.12)}$$

so then

$$\nabla^2(h\, r^{-1}) = -4\pi\, h(0)\, \delta(\mathbf{r}) + r^{-1}\,[h''(r) + (2/r)h'(r)\,] - 2\, r^{-2}\, h'(r)$$

$$= -4\pi\, h(0)\, \delta(\mathbf{r}) + h''(r)/\, r$$

which is the claim of (H.9). **QED**

Example of Theorem 2: $h(r) = e^{-jkr}$

$$h = e^{-jkr} \qquad h(0) = 1 \qquad h' = -jk\, e^{-jkr} \qquad h'' = -k^2\, e^{-jkr}$$

$$\nabla^2(e^{-jkr}/r) = -4\pi\, 1\, \delta(\mathbf{r}) + [-k^2\, e^{-jkr}\,] / r$$

$$= -4\pi\delta(r) - k^2(e^{-jkr}/r)$$

Thus,

$$(\nabla^2 + k^2)\,(e^{-jkr}/r) = -4\pi\delta(\mathbf{r}) \qquad \text{(H.13)}$$

or

$$-(\nabla^2 + k^2)\,(e^{-jkr}/4\pi r) = \delta(\mathbf{r}) \qquad \text{(H.14)}$$

If we translate the origin from **0** to **x'** and define $R = |\mathbf{r} - \mathbf{x}'|$ and rename **r** to be **x**, (H.13) becomes

$$-(\nabla^2 + k^2)\,(e^{-jkR}/4\pi R) = \delta(\mathbf{x}-\mathbf{x}') \qquad \text{(H.15)}$$

Thus we may identify $e^{-jkR}/4\pi R$ as the Helmholtz equation Green's Function. In the case k = 0 we get

$$-\nabla^2(1/4\pi R) = \delta(\mathbf{x}-\mathbf{x}') \qquad \text{(H.16)}$$

and we may identify $(1/4\pi R)$ as the Poisson equation Green's Function as shown in (A.0.3) and (A.0.4).

Theorem 3: $\nabla^2[\ln(1/r)] = -2\pi\delta(\mathbf{r})$ in two dimensions (H.1)'

Our 2D proof here will be an "copy, paste and edit" version of the 3D proof of Theorem 1. We shall use the word "volume" with symbol V to denote *area*. A "spherical cavity" now means a circular hole. It

should be understood that $\nabla^2 = \partial_x^2+\partial_y^2$ and $\delta(\mathbf{r})= \delta(x)\delta(y)$. We indicate all edits in red and equation numbers are the same as for Theorem 1 with a prime added.

Proof: Let volume V be all of 2D space. Carve out from V a small spherical cavity of radius a centered at $\mathbf{r} = 0$. If we call this spherical volume V_a and then $V' = V - V_a$ is the original volume with the spherical cavity. In order to show that some function $g(r) = \delta(\mathbf{r})$, one has to show that

$$\lim_{a\to 0} \int_{V'} dV\, g(r) = 0 \qquad \text{(H.2a)'}$$

$$\lim_{a\to 0} \int_{Va} dV\, g(r) = 1 \qquad \text{(H.2b)'}$$

This is basically the definition of $\delta(\mathbf{r})$. Since $\delta(\mathbf{r})$ has units L^{-2}, g(r) has units L^{-2}.

Comment: When any differential operator like ∇ or ∇^2 is applied to $\ln(r_0/r)$, the result is independent of r_0 so we can always take $r_0 = 1$. For example, $\partial_x [\ln(r_0/r)] = \partial_x [\ln r_0 + \ln(1/r)] = \partial_x \ln(1/r)$. In what follows, ln(r) and ln(1/r) are always acted upon by differential operators, so we can interpret these objects as dimensionless quantities $\ln(r/r_0)$ and $\ln(r_0/r)$. Then it is clear below that dim [g(r)] $= L^{-2}$.

Our candidate function of interest is

$$g(r) = -(1/2\pi)\, \nabla^2[\ln(1/r)] = +(1/2\pi)\, \nabla^2 [\ln(r)] \qquad \text{(H.3)'}$$

Using ∇^2 in polar coordinates acting on a function of r, one finds that, since $\partial_r(1) = 0$,

$$\nabla^2[\ln(r)] = (1/r)\partial_r(r\partial_r) [\ln(r)] = 0 \qquad r > 0 \qquad \text{(H.4)'}$$

so that

$$g(r) = -(1/2\pi)\, \nabla^2[\ln(1/r)] = 0 \qquad r > 0 \quad . \qquad \text{(H.5)'}$$

Thus, condition (H.2a)' is trivially satisfied since $r > 0$ everywhere in volume V'.

It remains to verify condition (H.2b)'. Consider the integral appearing in the left side of (H.2b)'

$$\int_{Va} dV\, g(r) = -(1/2\pi) \int_{Va} dV\, \nabla^2[\ln(1/r)] = -(1/2\pi) \int_{Va} dV\, \boldsymbol{\nabla} \bullet \boldsymbol{\nabla}[\ln(1/r)] \quad . \qquad \text{(H.6)'}$$

The divergence theorem says

$$\int_V dV\, \text{div}\, \mathbf{F} = \int_S d\mathbf{S} \bullet \mathbf{F} \qquad \textbf{(H.7)'}$$

where V is any closed volume whose surface is S, and d**S** points out. Using

$V = V_a$ and $\mathbf{F} = \nabla[\ln(1/r)] = \hat{\mathbf{r}}\partial_r(\ln(1/r)) = -\hat{\mathbf{r}}\partial_r(\ln r) = [-r^{-1}]\hat{\mathbf{r}}$

we find that [here $d\mathbf{S} = dS\,\hat{r}$ where $dS = ad\theta$ = a piece of circumference of the circle bounding V_a]

$$\text{LHS (H.7)} = \int_{V_a} dV\ \text{div}\ \nabla[\ln(1/r)] = \int_{V_a} dV\ \nabla^2[\ln(1/r)] = \int_{V_a} dV\ [-2\pi g(r)] = -2\pi \int_{V_a} dV\ g(r)$$

$$\text{RHS (H.7)} = \int_S d\mathbf{S} \bullet \nabla[\ln(1/r)] = \int d\theta\ [a\,\hat{\mathbf{r}}] \bullet \nabla[\ln(1/r)]|^{r=a} = \int d\theta\ [a\,\hat{\mathbf{r}}] \bullet [-a^{-1}\hat{\mathbf{r}}] = -2\pi$$

which tells us that $\int_{V_a} dV\ g(r) = 1$ for any a. Thus,

$$\lim_{a\to 0} \int_{V_a} dV\ g(r) = 1$$

and we have then verified (H.2b)'. Therefore we conclude that

$$-(1/2\pi)\nabla^2[\ln(1/r)] = \delta(\mathbf{r}) \qquad \text{(H.8)'}$$

or

$$\nabla^2[\ln(1/r)] = -2\pi\delta(\mathbf{r})$$

which is (H.1)'. **QED**