11.14.13

This is where I developed Appendix I which is now installed in lines doc. I split the 3D and 2D stuff into two separate appendices. This is "version 2" of an earlier draft of this appendix, don't know what was wrong with the first version.

Appendix I : The Poisson and Helmholtz Free-Space Propagators in 2D

Note: Appendix H deals with these propagators in 3D rather than 2D. Sections I.1 and I.2 below are basically "cut, paste and edit" versions of Sections H.1 and H.2, and we have made equation numbers match. However, Section I.3 is something new since it involves a "special function".

I.1 Overview and Meaning of Free-Space Propagators

This appendix proves two Facts: ($H_0^{(1)}$ is a Hankel function)

Fact 1 : $-\nabla^2[\ln(1/r)/2\pi] = \delta(\mathbf{r})$ *(I.2.1)* (I.1.1)

Fact 2 : $-(\nabla^2+k^2)[(j/4)\,H_0^{(1)}(kr)] = \delta(\mathbf{r})$ *(I.3.1)* (I.1.2)

Throughout this Appendix, ∇^2 is the usual 2D Laplacian operator,

$$\nabla^2 = \partial_x^2 + \partial_y^2. \qquad \text{and} \qquad \delta(\mathbf{r}) = \delta(x)\,\delta(y)\,. \tag{I.1.3}$$

In the two Facts above, if one replaces $\mathbf{r} \rightarrow \mathbf{r}\text{-}\mathbf{r}'$ (a simple translational shift of origin) ones finds

$$-\nabla^2[\ln(1/R)/2\pi] = \delta(\mathbf{r}\text{-}\mathbf{r}') \qquad R = |\,\mathbf{r} - \mathbf{r}'\,| \tag{I.1.4}$$

$$-(\nabla^2+k^2)[(j/4)\,H_0^{(1)}(kR)] = \delta(\mathbf{r}\text{-}\mathbf{r}') \qquad \delta(\mathbf{r}\text{-}\mathbf{r}') = \delta(x\text{-}x')\,\delta(y\text{-}y') \tag{I.1.5}$$

The quantities in brackets are known as free-space Green's Functions (Green Functions) or propagators, or as "fundamental solutions" :

$$\frac{1}{2\pi}\ln(1/R) = \text{the Poisson 2D free-space propagator} \tag{I.1.6}$$

$$(j/4)\,H_0^{(1)}(kR) = \text{the Helmholtz 2D free-space propagator} \tag{I.1.7}$$

The significance of these propagators is the following:

$$-\nabla^2 f(\mathbf{x}) = s(\mathbf{x}) \qquad \Rightarrow \qquad f(\mathbf{x}) = \int d^2x'\,[\ln(1/R)/2\pi]\,s(\mathbf{x}') + \text{homogeneous solutions}$$

The Poisson Equation (I.1.8)

$$-(\nabla^2+k^2)\, f(\mathbf{x}) = s(\mathbf{x}) \quad \Rightarrow \quad f(\mathbf{x}) = \int d^2x' \,[(j/4)\, H_0^{(1)}(kR)]\, s(\mathbf{x}') + \text{homogeneous solutions}$$

The Helmholtz Equation (I.1.9)

The equations on the left are inhomogeneous partial differential equations driven by source function $s(\mathbf{x})$. If one is careful to include in $s(\mathbf{x})$ all source contributions (such as those on boundary curves), one generally does not have to add any homogeneous solutions on the right. A homogeneous solution refers to $-\nabla^2 f_h(\mathbf{x}) = 0$, for example. The solutions shown on the right above can be instantly verified as follows:

$$f(\mathbf{x}) = \int d^2x' \,[\ln(1/R)/2\pi]\, s(\mathbf{x}') + f_h(\mathbf{x})$$

$$-\nabla^2 f(\mathbf{x}) = \int d^2x' \,(-\nabla^2 [\ln(1/R)/2\pi])\, s(\mathbf{x}') - \nabla^2 f_h(\mathbf{x}) = \int d^3x' \,\delta(\mathbf{r}-\mathbf{r}')\, s(\mathbf{x}') - 0 = s(\mathbf{x}) \qquad \text{(I.1.10)}$$

and similarly for $-(\nabla^2+k^2) f = g$.

A "free space" Green's Function g_F in general is a solution of

$$D\, g_F(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'), \qquad g_F(\mathbf{r}, \mathbf{r}') \to 0 \text{ as } \mathbf{r} \to \infty \qquad \text{(I.1.11)}$$

where D is some differential operator. The condition on the right says g_F must vanish on the Great Circle. More generally one can write

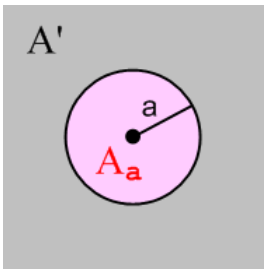
$$D\, g(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'), \qquad g(\mathbf{r}, \mathbf{r}') = 0 \text{ for } \mathbf{r} \text{ on some closed curve enclosing a region of interest} \qquad \text{(I.1.12)}$$

In this second form, the $\int d^2x'$ is over the area inside that closed surface. We shall not make use of this more general form in this document. George Green (1793-1841), by the way, was an English grain miller.

Looking at $f(\mathbf{x}) = \int d^2x' \,[\ln(1/R)/2\pi]\, s(\mathbf{x}') = \int g_F(\mathbf{x},\mathbf{x}')\, [s(\mathbf{x}')\, d^2x']$, one can say that the kernel Green's Function $g_F(\mathbf{x},\mathbf{x}')$ "propagates" a tiny piece of "source" $[s(\mathbf{x}')d^2x']$ from location $\mathbf{x}'$ to location $\mathbf{x}$ so that the solution $f(\mathbf{x})$ is then a sum of all such propagated contributions as the source ranges over the entire volume of interest, which for us is all 2D space where the source is non-vanishing.

I.2 Fact 1: $\nabla^2[\ln(1/r)] = -2\pi\delta(\mathbf{r})$ (I.2.1)

Proof: Let area A be all of 2D space. Cut out from A a small spherical hole of radius a centered at $\mathbf{r} = 0$. If we call this spherical area A_a and then $A' = A - A_a$ is the original area with the spherical hole cut out:



In order to show that some function g(**r**) = δ(**r**), one has to show that

$$\lim_{a\to 0} \int_{A'} dA\ g(\mathbf{r}) = 0 \qquad \text{(I.2.2a)}$$

$$\lim_{a\to 0} \int_{A_a} dA\ g(\mathbf{r}) = 1 \qquad \text{(I.2.2b)}$$

This is basically the definition of δ(**r**). Since δ(**r**) has units L^{-2}, g(r) has units L^{-2}.

Comment: When any differential operator like ∇ or ∇^2 is applied to $\ln(r_0/r)$, the result is independent of r_0 so we can always take $r_0 = 1$. For example, $\partial_x [\ln(r_0/r)] = \partial_x [\ln r_0 + \ln(1/r)] = \partial_x \ln(1/r)$. In what follows, ln(r) and ln(1/r) are always acted upon by differential operators, so we can interpret these objects as dimensionless quantities $\ln(r/r_0)$ and $\ln(r_0/r)$ for any r_0. Then it is clear below that dim [g(r)] = L^{-2}.

Our candidate function of interest is

$$g(r) = -(1/2\pi)\,\nabla^2[\ln(1/r)] = +(1/2\pi)\,\nabla^2[\ln(r)] \ . \qquad \text{(I.2.3)}$$

Using ∇^2 in polar (cylindrical without the z) coordinates acting on a function of r, one finds that, since $\partial_r(1) = 0$,

$$\nabla^2[\ln(r)] = (1/r)\partial_r(r\partial_r)[\ln(r)] = 0 \qquad r > 0 \qquad \text{(I.2.4)}$$

so that

$$g(r) = -(1/2\pi)\,\nabla^2[\ln(1/r)] = 0 \qquad r > 0 \ . \qquad \text{(I.2.5)}$$

Thus, condition (I.2.2a) is trivially satisfied since r > 0 everywhere in area A' for any a > 0.

It remains to verify condition (I.2.2b). Consider the integral appearing in the left side of (I.2.2b)

$$\int_{A_a} dA\ g(r) = -(1/4\pi) \int_{A_a} dA\ \nabla^2[1/r] = -(1/4\pi) \int_{A_a} dA\ \nabla \bullet \nabla[1/r] \ . \qquad \text{(I.2.6)}$$

The divergence Fact in 2D says

$$\int_A dA \text{ div } \mathbf{F} = \oint_C d\mathbf{s} \bullet \mathbf{F} \qquad \textbf{(I.2.7)}$$

where A is any closed area whose bounding curve is C, and where $d\mathbf{s} = ds\ \hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is normal to C at any given point on C. Notice that this closed area is necessarily planar since everything is 2D here. Using

$$A = A_a = \text{disk of radius a} \quad \text{and} \quad \mathbf{F} = \nabla[\ln(1/r)] = \hat{\mathbf{r}}\partial_r(\ln(1/r)) = -\hat{\mathbf{r}}\partial_r(\ln r) = [-r^{-1}]\hat{\mathbf{r}}$$

we find that

$$\text{LHS (I.2.7)} = \int_{A_a} dA \text{ div } \nabla[\ln(1/r)] = \int_{A_a} dA\ \nabla^2[\ln(1/r)] = \int_{A_a} dA\ [-2\pi g(r)] = -2\pi \int_{A_a} dA\ dA\ g(r)$$

$$\text{RHS (I.2.7)} = \int_C d\mathbf{s} \bullet \nabla[\ln(1/r)] = \int [a d\theta\ \hat{\mathbf{r}}] \bullet \nabla[\ln(1/r)]|^{r=a} = \int d\theta\ [a\hat{\mathbf{r}}] \bullet [-a^{-1}\hat{\mathbf{r}}] = -2\pi$$

which tells us that $\int_{Aa} dA\ g(r) = 1$ for any a. Thus,

$$\lim_{a\to 0} \int_{Aa} dA\ g(r) = 1$$

and we have then verified (I.2.2b). Therefore we conclude that the candidate $g(\mathbf{r})$ of (I.2.3) is in fact the same as $\delta(\mathbf{r})$ so

$$-(1/2\pi)\nabla^2[\ln(1/r)] = \delta(\mathbf{r}) \qquad \text{(I.2.8)}$$

or

$$\nabla^2[\ln(1/r)] = -2\pi\delta(\mathbf{r}) \qquad \text{(I.2.9)}$$

which is (I.2.1). **QED**

I.3 Fact 2: $-(\nabla^2+k^2)[(j/4)H_0^{(1)}(kr)] = \delta(\mathbf{r})$ (I.3.1)

We seek the solution E(r) of this equation

$$-(\nabla^2+k^2)E(r) = \delta(\mathbf{r}) \qquad \text{where} \quad E(r\to\infty) = 0 \qquad \text{(I.3.2)}$$

which we write as

$$\nabla^2 E + k^2 E = -\delta(\mathbf{r}).$$

Using polar coordinates this says

$$r^{-1}\partial_r(r\partial_r E) + k^2 E = -\delta(\mathbf{r})$$

or

$$E'' + r^{-1}E' + k^2E = -\delta(\mathbf{r})$$

or

$$r^2E''(r) + rE'(r) + r^2k^2E(r) = -\delta(\mathbf{r}) \quad . \qquad \text{(I.3.3)}$$

Writing $E(r) = F(kr)$ we get

$$r^2k^2F''(kr) + rk\,F'(kr) + r^2k^2F(kr) = -\delta(\mathbf{r})$$

or

$$(rk)^2F''(kr) + (rk)\,F'(kr) + (rk)^2F(kr) = -\delta(\mathbf{r})$$

or

$$z^2F''(z) + z\,F'(z) + z^2F(z) = -\delta(\mathbf{r}) \qquad \text{where } z = kr \quad . \qquad \text{(I.3.4)}$$

Away from r = z = 0, this is Bessel's equation of index 0 (A&S 10.2.1) so solutions are Bessel functions like these

$$F(z) = J_0(z),\ Y_0(z),\ H_0^{(1)}(z),\ H_0^{(2)}(z). \qquad z = kr \qquad \text{(I.3.5)}$$

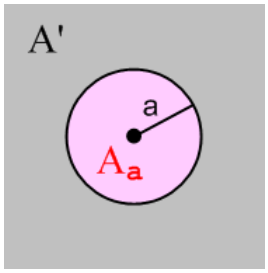
which are Bessel functions of the first, second and third kind. The third kind functions (the H's) are called Hankel Functions. If we assume that k has a tiny positive imaginary part (see Comments later), then of all the functions just listed, only $H_0^{(1)}(kr)$ has decaying behavior for large r (A&S 10.2.5). We therefore put forward the following *candidate* for a delta function

$$g(r) = -(\nabla^2 + k^2)\,C\,H_0^{(1)}(kr) \quad . \qquad \text{(I.3.6)}$$

Recall from Section I.2 that a successful $\delta(\mathbf{r})$ candidate must satisfy these two conditions

$$\lim_{a\to 0} \int_{A'} dA\ g(\mathbf{r}) = 0 \qquad \textit{(I.2.2a)}$$

$$\lim_{a\to 0} \int_{A_a} dA\ g(\mathbf{r}) = 1 \qquad \textit{(I.2.2b)}$$



Our candidate g(r) vanishes within any region A' no matter how small the hole because g(r) = 0 for any r> 0, so the first condition is already met. It remains only to show that the second condition is also met. We must then show that

$$\lim_{a\to 0} \int_{A_a} dA \{- (\nabla^2+k^2) C H_0^{(1)}(kr)\} = 1 . \qquad (I.3.7)$$

Since A_a is a very small disk as we approach the limit, we may use the small argument behavior of our candidate g(r) in studying the situation. We know that

$$H_0^{(1)}(kr) \approx (2j/\pi) \ln(kr) \qquad // \text{ A\&S } 10.7.2 \qquad (I.3.8)$$

so what we need to show is that

$$\lim_{a\to 0} \int_{A_a} dA \{- (\nabla^2+k^2) C (2j/\pi) \ln(kr)\} = 1$$

or

$$- C (2j/\pi) \lim_{a\to 0} \int_{A_a} dA \{ (\nabla^2+k^2) \ln(kr)\} = 1$$

or

$$- C (2j/\pi)2\pi \lim_{a\to 0} \int_0^a rdr\{ (\nabla^2+k^2) \ln(kr)\} = 1 \qquad // \int d\theta = 2\pi$$

or

$$C (4/j) \lim_{a\to 0} \int_0^a rdr\{ (\nabla^2+k^2) \ln(kr)\} = 1 . \qquad (I.3.9)$$

Now consider :

$$\lim_{a\to 0} [\int_0^a rdr \ln(kr)] = \lim_{a\to 0} [(1/4)a^2\{2\ln(ka)-1\}] = 0 . \qquad (I.3.10)$$

Thus, the $k^2 \ln(kr)$ term in (I.3.9) makes no contribution in the limit, so we then have to show that

$$C (4/j) \lim_{a\to 0} \int_0^a rdr \nabla^2 [\ln(kr)] = 1 . \qquad (I.3.11)$$

But (I.2.9) says that

$$\nabla^2[\ln(r)] = 2\pi\delta(\mathbf{r}) . \qquad \textit{(I.2.9)}$$

Now

$$\delta(\mathbf{r}) = \delta(x)\delta(y) = \delta(r)/2\pi r \qquad (I.3.12)$$

since

$$1 = \int\int dxdy\, \delta(x)\delta(y) = \int rdr \int d\theta\, \delta(r)/2\pi r = 2\pi \int rdr\, \delta(r)/2\pi r = \int dr\, \delta(r) = 1 .$$

Therefore

$$\nabla^2[\ln(r)] = \delta(r)/r \qquad \text{(I.3.13)}$$

and then

$$\nabla^2[\ln(kr)] = \nabla^2[\ln(k) + \ln(r)] = \nabla^2[\ln(r)] = \delta(r)/r \ . \qquad \text{(I.3.14)}$$

Inserting this last result into (I.3.11) then gives

$$C\ (4/j)\ \lim_{a\to 0} \int_0^a r dr\ \nabla^2\ [\ \ln(kr)]\ = 1$$

$$C\ (4/j)\ \lim_{a\to 0} \int_0^a r dr\ \delta(r)/r\ = 1$$

$$C\ (4/j)\ \lim_{a\to 0} \int_0^a dr\ \delta(r)\ = 1$$

$$C\ (4/j)\ \lim_{a\to 0} 1\ = 1$$

$$C\ (4/j)\ = 1\ .$$

Thus, we have a solution if we select constant C = (j/4). Therefore, the solution to (I.3.2) is

$$E(r) = C\ H_0^{(1)}(kr) = (j/4)\ H_0^{(1)}(kr)\ . \qquad \text{(I.3.15)}$$

Stakgold Vol II page 55 (5.120) confirms this result where $\sqrt{\lambda}\ = k$.

Therefore we have shown that

$$-\ (\nabla^2+k^2)\ [(j/4)\ H_0^{(1)}(kr)]\ = \delta(\mathbf{r}) \qquad \text{(I.3.16)}$$

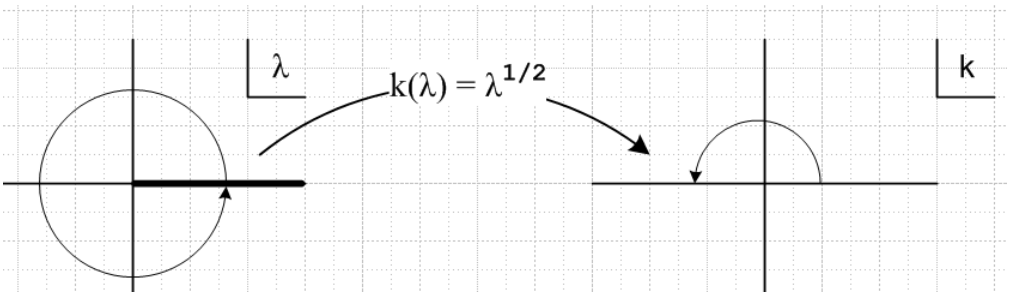
which is the Fact stated as (I.3.1). **QED**

On page 54 Stakgold gives the solution to $-\ (\nabla^2+k^2\ )\ E(r) = \delta(\mathbf{r})$ for $n\geq 2$ dimensions as (5.118):

$$E_n(r;\lambda) = \frac{i}{4}\left(\frac{\sqrt{\lambda}}{2\pi r}\right)^{(n-2)/2} H^{(1)}_{(n/2)-1}(\sqrt{\lambda}r), \qquad n \geq 2. \qquad (5.118)$$

Comments:

1. Complex Helmholtz Parameter. Stakgold considers the Helmholtz operator to be λ which is our k^2. He regards λ as a complex variable which can lie anywhere in the complex λ plane. If we consider the function $k(\lambda) = \lambda^{1/2}$, we find that it has a branch point at $\lambda = 0$. If we take the branch cut to the right, then one of the two Riemann sheets in λ-space for this function maps to the upper half k-plane as shown. This is the branch of $\lambda^{1/2}$ that Stakgold selects and that is why we think of k and therefore k^2 as having a tiny positive imaginary part. The point is that we approach the positive real axis from above, not from below. It is this assumption that causes the large-r-decaying solution to our problem to be $H_0^{(1)}(kr)$ instead of $H_0^{(2)}(kr)$.



2. Helmholtz morphs into Poisson. We have shown that

$$-(\nabla^2+k^2)\,[(j/4)\,H_0^{(1)}(kr)] = \delta(\mathbf{r}) \quad . \qquad \textit{(I.3.16)}$$

In the limit that $k << 1$, we showed above that

$$H_0^{(1)}(kr) \approx (2j/\pi)\ln(kr) \qquad \text{// A\&S 10.7.2} \qquad \textit{(I.3.8)}$$

In this limit we then have

$$-(\nabla^2+k^2)\,[(j/4))\,(2j/\pi)\ln(kr)\,] = \delta(\mathbf{r})$$

or

$$-(\nabla^2)\,[(1/2\pi)\ln(kr)\,] = \delta(\mathbf{r})$$

and this is in agreement with the Poisson result (I.1.1). So as the Helmholtz equation morphs into the Poisson equation as $k \rightarrow 0$, the Helmholtz propagator morphs into the Poisson propagator.