11.14.13

This is the "version 1" draft of Appendix I, it got replaced right away by Version 2 for some reason. In version 2 theorems become facts, and there is some reorganization.

Appendix I : The Poisson and Helmholtz Free-Space Propagators in 2D

Note: This Appendix was constructed from Appendix H using the time-honored method of "copy, paste and edit".

I.1 Overview and Meaning of Free-Space Propagators

This appendix contains two theorems and an application of the second theorem:

Theorem : $-\nabla^2[1/4\pi r] = \delta(\mathbf{r})$ *(I.2.1)* (I.1.1)

Theorem : $-\nabla^2[h(r)/r] = 4\pi\, h(0)\, \delta(\mathbf{r}) - h''(r)/r$ *(I.3.1)* (I.1.2)

Application of Theorem B : $-(\nabla^2+k^2)(e^{-jkr}/4\pi r) = \delta(\mathbf{r})$ *(I.3.5)* (I.1.3)

Throughout, ∇^2 is the usual 2D Laplacian operator $\nabla^2 = \partial_x^2 + \partial_y^2$. In the first and last results above, if one replaces $\mathbf{r} \rightarrow \mathbf{r}-\mathbf{r}'$ (a simple translational shift of origin) ones finds

$-\nabla^2[1/4\pi R] = \delta(\mathbf{r}-\mathbf{r}')$ $\quad R = |\,\mathbf{r}-\mathbf{r}'\,|$ (I.1.4)

$-(\nabla^2+k^2)[e^{-jkR}/4\pi R] = \delta(\mathbf{r}-\mathbf{r}')$ $\quad \delta(\mathbf{r}-\mathbf{r}') = \delta(x-x')\,\delta(y-y')\,\delta(z-z')$ (I.1.5)

The quantities in brackets are known as free-space Green's Functions (Green Functions) or propagators, or as "fundamental solutions":

$1/4\pi R$ = the Poisson 2D free-space propagator (I.1.6)

$e^{-jkR}/4\pi R$ = the Helmholtz 2D free-space propagator (I.1.7)

The significance of these propagators is the following:

$-\nabla^2 f(\mathbf{x}) = s(\mathbf{x})$ $\Rightarrow$ $f(\mathbf{x}) = \int d^2x'\,[1/4\pi R]\, s(\mathbf{x}')$ + homogeneous solutions

The Poisson Equation (I.1.8)

$-(\nabla^2+k^2) f(\mathbf{x}) = s(\mathbf{x})$ $\Rightarrow$ $f(\mathbf{x}) = \int d^2x'\,[e^{-jkR}/4\pi R]\, s(\mathbf{x}')$ + homogeneous solutions

The Helmholtz Equation (I.1.9)

The equations on the left are inhomogeneous partial differential equations driven by source function $s(\mathbf{x})$. If one is careful to include in $s(\mathbf{x})$ all source contributions (such as those on boundary surfaces), one generally does not have to add any homogeneous solutions on the right. A homogeneous solution refers to $-\nabla^2 f_h(\mathbf{x}) = 0$, for example. The solutions shown on the right above can be instantly verified as follows:

$$f(\mathbf{x}) = \int d^2x' \, [1/4\pi R] \, s(\mathbf{x}') + f_h(\mathbf{x})$$

$$-\nabla^2 f(\mathbf{x}) = \int d^2x' \, (-\nabla^2 \, [1/4\pi R]\,) \, s(\mathbf{x}') - \nabla^2 f_h(\mathbf{x}) = \int d^3x' \, \delta(\mathbf{r}-\mathbf{r}') \, s(\mathbf{x}') - 0 = s(\mathbf{x}) \qquad \text{(I.1.10)}$$

and similarly for $-(\nabla^2+k^2) f = g$.

A "free space" Green's Function g_F in general is a solution of

$$D \, g_F(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'), \qquad g_F(\mathbf{r}, \mathbf{r}') \to 0 \text{ as } \mathbf{r} \to \infty \qquad \text{(I.1.11)}$$

where D is some differential operator. The condition on the right says g_F must vanish on the Great Circle. More generally one can write

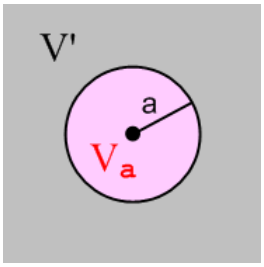
$$D \, g(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'), \qquad g(\mathbf{r}, \mathbf{r}') = 0 \text{ for } \mathbf{r} \text{ on some closed curve enclosing a region of interest} \qquad \text{(I.1.12)}$$

In this second form, the $\int d^2x'$ is over the area inside that closed surface. We shall not make use of this more general form in this document. George Green (1793-1841), by the way, was an English grain miller.

Looking at $f(\mathbf{x}) = \int d^2x' \, [1/4\pi R] \, s(\mathbf{x}') = \int g_F(\mathbf{x},\mathbf{x}') \, [s(\mathbf{x}') \, d^2x']$, one can say that the kernel Green's Function $g_F(\mathbf{x},\mathbf{x}')$ "propagates" a tiny piece of "source" $[s(\mathbf{x}')d^2x']$ from location $\mathbf{x}'$ to location $\mathbf{x}$ so that the solution $f(\mathbf{x})$ is then a sum of all such propagated contributions as the source ranges over the entire volume of interest, which for us is all 2D space where the source is non-vanishing.

H.2 Theorem: $\qquad -\nabla^2[1/r] = 4\pi\delta(\mathbf{r}) \qquad \text{(I.2.1)}$

Proof: Let volume V be all of 2D space. Carve out from V a small spherical hole of radius a centered at $\mathbf{r} = 0$. If we call this spherical area V_a and then $V' = V - V_a$ is the original area with the spherical hole carved out:



In order to show that some function $g(\mathbf{r}) = \delta(\mathbf{r})$, one has to show that

$$\lim_{a\to 0} \int_{V'} dV\, g(\mathbf{r}) = 0 \qquad \text{(I.2.2a)}$$

$$\lim_{a\to 0} \int_{V_a} dV\, g(\mathbf{r}) = 1 \qquad \text{(I.2.2b)}$$

This is basically the definition of $\delta(\mathbf{r})$. Since $\delta(\mathbf{r})$ has units L^{-2}, g(r) has units L^{-2}.

Comment: When any differential operator like ∇ or ∇^2 is applied to $\ln(r_0/r)$, the result is independent of r_0 so we can always take $r_0 = 1$. For example, $\partial_x [\ln(r_0/r)] = \partial_x [\ln r_0 + \ln(1/r)] = \partial_x \ln(1/r)$. In what follows, ln(r) and ln(1/r) are always acted upon by differential operators, so we can interpret these objects as dimensionless quantities $\ln(r/r_0)$ and $\ln(r_0/r)$. Then it is clear below that dim [g(r)] $= L^{-2}$.

Our candidate function of interest is

$$g(r) = -(1/2\pi)\,\nabla^2[\ln(1/r)] = +(1/2\pi)\,\nabla^2[\ln(r)] \ . \qquad \text{(I.2.3)}$$

Using ∇^2 in polar (cylindrical) coordinates acting on a function of r, one finds that, since $\partial_r(1) = 0$,

$$\nabla^2[\ln(r)] = (1/r)\partial_r(r\partial_r)[\ln(r)] = 0 \qquad r > 0 \qquad \text{(I.2.4)}$$

so that

$$g(r) = -(1/2\pi)\,\nabla^2[\ln(1/r)] = 0 \qquad r > 0 \ . \qquad \text{(I.2.5)}$$

Thus, condition (I.2.2a) is trivially satisfied since $r > 0$ everywhere in area V'.

It remains to verify condition (I.2.2b). Consider the integral appearing in the left side of (I.2.2b)

$$\int_{V_a} dV\, g(r) = -(1/4\pi) \int_{V_a} dV\, \nabla^2[1/r] = -(1/4\pi) \int_{V_a} dV\, \boldsymbol{\nabla} \bullet \boldsymbol{\nabla}[1/r] \ . \qquad \text{(I.2.6)}$$

The divergence theorem says

$$\int_V dV\, \mathrm{div}\, \mathbf{F} = \int_S d\mathbf{S} \bullet \mathbf{F} \qquad \textbf{(I.2.7)}$$

where V is any closed area whose bounding curve is S, and d**S** points out. Using

$$V = V_a \qquad \text{and} \qquad \mathbf{F} = \boldsymbol{\nabla}[\ln(1/r)] = \hat{\mathbf{r}}\partial_r(\ln(1/r)) = -\hat{\mathbf{r}}\partial_r(\ln r) = [-r^{-1}]\,\hat{\mathbf{r}}$$

we find that

LHS (I.2.7) $= \int_{V_a} dV \; \text{div}\, \nabla[\ln(1/r)] = \int_{V_a} dV \; \nabla^2[\ln(1/r)] = \int_{V_a} dV \; [-2\pi g(r)] = -2\pi \int_{V_a} dV \; g(r)$

RHS (I.2.7) $= \int_S d\mathbf{S} \bullet \nabla[\ln(1/r)] = \int d\theta \; [a\,\hat{\mathbf{r}}] \bullet \nabla[\ln(1/r)]|^{r=a} = \int d\theta \; [a\,\hat{\mathbf{r}}] \bullet [-a^{-1}\hat{\mathbf{r}}] = -2\pi$

which tells us that $\int_{V_a} dV \; g(r) = 1$ for any a. Thus,

$\lim_{a \to 0} \int_{V_a} dV \; g(r) = 1$

and we have then verified (I.2.2b). Therefore we conclude that the candidate $g(\mathbf{r})$ of (I.2.3) is in fact the same as $\delta(\mathbf{r})$ so

$$-(1/2\pi)\nabla^2[\ln(1/r)] = \delta(\mathbf{r}) \qquad \text{(I.2.8)}$$

or

$$\nabla^2[\ln(1/r)] = -2\pi\delta(\mathbf{r}) \qquad \text{(I.2.9)}$$

which is (I.2.1). **QED**

H.3 Theorem: $\nabla^2[h(r)/r] = -4\pi\, h(0)\, \delta(\mathbf{r}) + h''(r)/r$ (I.3.1)

Proof: Start with this vector identity, which is valid in 2D as well as 3D,

$$\nabla^2(\varphi\psi) = \varphi\nabla^2\psi + \psi\nabla^2\varphi + 2\,\nabla\varphi \bullet \nabla\psi \qquad \text{(I.3.2)}$$

This identity is valid in any number of dimensions (implied sum on i from 1 to N) ,

$$\partial_i^2(\varphi\psi) = \partial_i[(\partial_i\varphi)\psi + \psi(\partial_i\varphi)] = (\partial_i^2\varphi)\psi + (\partial_i\varphi)(\partial_i\psi) + \varphi(\partial_i^2\psi) + (\partial_i\varphi)(\partial_i\psi) .$$

We insert this small fact to be used below (cylindrical coordinates)

$$\nabla \ln(1/r) = \nabla[-\ln(r)] = \hat{\mathbf{r}}\,\partial_r[-\ln(r)] = -(1/r)\,\hat{\mathbf{r}}$$

So apply (I.3.2) to the case $\varphi = h$ and $\psi = \ln(1/r)$,

$$\nabla^2(h\ln(1/r)) = h\nabla^2(\ln(1/r)) + \ln(1/r)\,\nabla^2 h + 2\,\nabla h \bullet \nabla \ln(1/r)$$

$$= -h\,2\pi\,\delta(\mathbf{r}) + \ln(1/r)\nabla^2 h + 2\,[\,h'\,\hat{\mathbf{r}} \bullet (-r^{-1})\,\hat{\mathbf{r}}\,] \qquad \text{// using (I.2.9) and fact just above}$$

$$= -2\pi\,h(0)\,\delta(\mathbf{r}) + \ln(1/r)\nabla^2 h - 2\,r^{-1}\,h'(r) \; . \qquad \text{(I.3.3)}$$

Algebra shows that, using polar (cylindrical) coordinates,

$$\nabla^2 h = (1/r)\partial_r(r\partial_r)h(r) = h''(r) + r^{-1}\ h'(r) \qquad \text{(I.3.4)}$$

so then

$$\nabla^2(h\ \ln(1/r)) = -\ 2\pi\ h(0)\ \delta(\mathbf{r}) + \ln(1/r)\ [h''(r) + r^{-1}h'(r)\ ] - 2\ r^{-1}\ h'(r)$$

$$= -\ 2\pi\ h(0)\ \delta(\mathbf{r}) + \ln(1/r)h''(r) + [\ \ln(1/r) - 2]\ r^{-1}\ h'(r)$$

which is the claim of (I.3.1). **QED**

Application: Let $h(r) = e^{-jkr}$

$$h = e^{-jkr} \qquad h(0) = 1 \qquad h' = -jk\ e^{-jkr} \qquad h'' = -k^2\ e^{-jkr}$$

$$\nabla^2[e^{-jkr}\ \ln(1/r)\ ] = -\ 2\pi\ \delta(\mathbf{r}) + \ln(1/r)\ [-k^2\ e^{-jkr}] + [\ \ln(1/r) - 2]\ r^{-1}\ [-jk\ e^{-jkr}]$$

$$= -\ 2\pi\ \delta(\mathbf{r}) + e^{-jkr}\ \{\ -k^2\ \ln(1/r)\ -jk\ r^{-1}\ [\ \ln(1/r) - 2]\ \}$$

Thus,

$$(\ \nabla^2+k^2)\ [e^{-jkr}\ \ln(1/r)\ ] = -\ 2\pi\delta(\mathbf{r}) \quad -jk\ r^{-1}\ [\ \ln(1/r) - 2]\ e^{-jkr}$$

I was not expecting this mess!! What does Stak say?

or

$$-\ (\nabla^2+k^2)\ (e^{-jkr}/4\pi r) = \delta(\mathbf{r}) \qquad \text{(I.3.5)}$$

Theorem 3: $\nabla^2[\ln(1/r)] = -\ 2\pi\delta(\mathbf{r})$ in two dimensions (I.1)'

Our 2D proof here will be an "copy, paste and edit" version of the 3D proof of Theorem 1. We shall use the word "volume" with symbol V to denote *area*. A "spherical cavity" now means a circular hole. It should be understood that $\nabla^2 = \partial_x^2+\partial_y^2$ and $\delta(\mathbf{r}) = \delta(x)\delta(y)$. We indicate all edits in red and equation numbers are the same as for Theorem 1 with a prime added.

Proof: Let volume V be all of 2D space. Carve out from V a small spherical cavity of radius a centered at $\mathbf{r} = 0$. If we call this spherical volume V_a and then $V' = V - V_a$ is the original volume with the spherical cavity. In order to show that some function $g(r) = \delta(\mathbf{r})$, one has to show that

$$\lim_{a\to 0} \int_{V'} dV\ g(r) = 0 \qquad \text{(I.2a)}'$$

$$\lim_{a\to 0} \int_{Va} dV\ g(r) = 1 \qquad \text{(I.2b)}'$$

This is basically the definition of $\delta(\mathbf{r})$. Since $\delta(\mathbf{r})$ has units L^{-2}, g(r) has units L^{-2}.

Comment: When any differential operator like ∇ or ∇^2 is applied to $\ln(r_0/r)$, the result is independent of r_0 so we can always take $r_0 = 1$. For example, $\partial_x [\ln(r_0/r)] = \partial_x [\ln r_0 + \ln(1/r)] = \partial_x \ln(1/r)$. In what follows, ln(r) and ln(1/r) are always acted upon by differential operators, so we can interpret these objects as dimensionless quantities $\ln(r/r_0)$ and $\ln(r_0/r)$. Then it is clear below that dim [g(r)] $= L^{-2}$.

Our candidate function of interest is

$$g(r) = -(1/2\pi)\,\nabla^2[\ln(1/r)] = +(1/2\pi)\,\nabla^2[\ln(r)] \qquad \text{(I.3)'}$$

Using ∇^2 in polar coordinates acting on a function of r, one finds that, since $\partial_r(1) = 0$,

$$\nabla^2[\ln(r)] = (1/r)\partial_r(r\partial_r)[\ln(r)] = 0 \qquad r > 0 \qquad \text{(I.4)'}$$

so that

$$g(r) = -(1/2\pi)\,\nabla^2[\ln(1/r)] = 0 \qquad r > 0 \quad . \qquad \text{(I.5)'}$$

Thus, condition (I.2a)' is trivially satisfied since $r > 0$ everywhere in volume V'.

It remains to verify condition (I.2b)'. Consider the integral appearing in the left side of (I.2b)'

$$\int_{V_a} dV\, g(r) = -(1/2\pi) \int_{V_a} dV\, \nabla^2[\ln(1/r)] = -(1/2\pi) \int_{V_a} dV\, \boldsymbol{\nabla} \bullet \boldsymbol{\nabla}[\ln(1/r)] \quad . \qquad \text{(I.6)'}$$

The divergence theorem says

$$\int_V dV\, \mathrm{div}\, \mathbf{F} = \int_S d\mathbf{S} \bullet \mathbf{F} \qquad \textbf{(I.7)'}$$

where V is any closed volume whose surface is S, and d**S** points out. Using

$$V = V_a \qquad \text{and} \qquad \mathbf{F} = \boldsymbol{\nabla}[\ln(1/r)] = \hat{\mathbf{r}}\partial_r(\ln(1/r)) = -\hat{\mathbf{r}}\partial_r(\ln r) = [-r^{-1}]\,\hat{\mathbf{r}}$$

we find that [here $d\mathbf{S} = dS\,\hat{r}$ where $dS = a d\theta$ = a piece of circumference of the circle bounding V_a]

$$\text{LHS (I.7)} = \int_{V_a} dV\, \mathrm{div}\, \boldsymbol{\nabla}[\ln(1/r)] = \int_{V_a} dV\, \nabla^2[\ln(1/r)] = \int_{V_a} dV\, [-2\pi g(r)] = -2\pi \int_{V_a} dV\, g(r)$$

$$\text{RHS (I.7)} = \int_S d\mathbf{S} \bullet \boldsymbol{\nabla}[\ln(1/r)] = \int d\theta\, [a\,\hat{\mathbf{r}}] \bullet \boldsymbol{\nabla}[\ln(1/r)]|^{r=a} = \int d\theta\, [a\,\hat{\mathbf{r}}] \bullet [-a^{-1}\hat{\mathbf{r}}] = -2\pi$$

which tells us that $\int_{V_a}$ dV g(r) = 1 for any a. Thus,

$$\lim_{a\to 0} \int_{V_a} dV\, g(r) = 1$$

and we have then verified (I.2b)'. Therefore we conclude that

$$-(1/2\pi)\nabla^2[\ln(1/r)] = \delta(\mathbf{r}) \qquad \text{(I.8)'}$$

or

$$\nabla^2[\ln(1/r)] = -2\pi\delta(\mathbf{r})$$

which is (I.1)'. **QED**