This is where I first wrote up this appendix, it is now all installed.

Appendix J : The 3D→2D Propagator Transition

Infinitely long transmission lines are -- in the transmission line limit of long wavelength -- basically 2D objects rather than 3D objects. We see that fact appearing in various Chapters and Appendices of this document. Here we wish to focus on this single fact.

Case 1

In Chapter 1 we presented the natural 3D view of transmission lines with equations like the following taken from (1.5.3), (1.5.4) and (1.5.23), where we used the King gauge,

$$(\nabla^2 + \beta^2)\varphi = -(1/\varepsilon)\,\Sigma_i \rho_i \quad \Leftrightarrow \quad \varphi(\mathbf{x},\omega) = \frac{1}{4\pi\xi}\,\Sigma_i \int \rho_{ci}(\mathbf{x}',\omega)\frac{e^{-j\beta R}}{R}\,dV'$$

$$(\nabla^2 + \beta^2)\mathbf{A} = -\Sigma_i \mu_i \mathbf{J}_i \quad \Leftrightarrow \quad \mathbf{A}(\mathbf{x},\omega) = \frac{1}{4\pi}\,\Sigma_i \int \mu_i \mathbf{J}_i(\mathbf{x}',\omega)\frac{e^{-j\beta R}}{R}\,dV' \ , \qquad (J.1)$$

The Helmholtz integrals on the right are particular solutions of the PDEs on the left. The equations on the right are derived from those on the left as shown in Appendix H where we had the more generic statement that

$$-(\nabla^2+k^2)\,f(\mathbf{x}) = s(\mathbf{x}) \quad \Rightarrow \quad f(\mathbf{x}) = \int d^3x'\,[e^{-jkR}/4\pi R]\,s(\mathbf{x}') \ + \text{homogeneous solutions}$$

The 3D Helmholtz Equation — particular solution *(H.1.9)*

The object $[e^{-jkR}/4\pi R]$ is the 3D free-space Helmholtz propagator as discussed in Appendix H.

If it happens that $f(\mathbf{x}) = f(x,y)$ in this last equation, then $\partial_z^2 f = 0$ and we find ourselves looking at a 2D Helmholtz equation which has a completely different-looking particular solution, where $\nabla^2 = \nabla_{2D}^2 + \partial_z^2$,

$$-(\nabla_{2D}^2+k^2)\,f(\mathbf{x}) = s(\mathbf{x}) \quad \Rightarrow \quad f(\mathbf{x}) = \int d^2x'\,[(j/4)\,H_0^{(1)}(kR)]\,s(\mathbf{x}') \ + \text{homogeneous solutions}$$

The 2D Helmholtz Equation — particular solution *(I.1.9)*

This is the most abrupt and simple way the transition from 3D to 2D can occur.

If k is small, meaning the corresponding wavelength $\lambda = 2\pi/k$ is large, we can take the small k limit of the above two particular integrals. The limit of $[e^{-jkR}/4\pi R]$ is completely obvious, whereas the limit of the 2D propagator $[(j/4)\,H_0^{(1)}(kR)]$ is less obvious:

$H_0^{(1)}(kr) \approx (2j/\pi) \ln(kr)$ // NIST 10.7.2 *(I.3.8)*

so that

$$[(j/4)\, H_0^{(1)}(kR)] \approx -(1/2\pi)\ln(kR) = [-(1/2\pi)\ln(R)] - (1/2\pi)\ln(k)\,. \qquad (J.2)$$

If we momentarily ignore the inconvenient constant $-(1/2\pi)\ln(k)$, we can say that

$$\text{2D Helmholtz propagator} = [\tfrac{j}{4}\, H_0^{(1)}(kR)] \rightarrow [-\tfrac{1}{2\pi}\ln(R)] = [-\tfrac{1}{4\pi}\ln(R^2)] = [\tfrac{1}{2\pi}\ln(1/R)]$$

The objects on the right are in fact the 2D Poisson propagator which belongs to this pair of PDE's and their particular solutions,

$-\nabla^2 f(\mathbf{x}) = s(\mathbf{x}) \quad \Rightarrow \quad f(\mathbf{x}) = \int d^3x'\, [1/4\pi R]\, s(\mathbf{x}') + \text{homogeneous solutions}$

The 3D Poisson Equation particular solution *(H.1.8)*

$-\nabla_{2D}^2 f(\mathbf{x}) = s(\mathbf{x}) \quad \Rightarrow \quad f(\mathbf{x}) = \int d^2x'\, [\ln(1/R)/2\pi]\, s(\mathbf{x}') + \text{homogeneous solutions}$

The 2D Poisson Equation particular solution *(I.1.8)*

Since in our applications $f(\mathbf{x})$ is always a potential like φ or $\mathbf{A}$, and since

$\mathbf{B} = \text{curl}\ \mathbf{A} \qquad \mathbf{E} = -\text{grad}\ \varphi - \partial_t \mathbf{A}$ *(1.3.1)*

we see that a constant like $-(1/2\pi)\ln(k)$ added to a potential has no effect on the physical fields **E** and **B**, so we can just ignore such constants. Another way to say this is that the zero level of a potential is always arbitrary so additive constants are meaningless. In Chapter 4 we are only really concerned with the potential *difference* V(z) or W(z) between conductors.

We can now look at some of the 3D/2D "transitions" that occurred in other parts of the document.

Case 2

In Section 4.4 we had

$$V(z) \equiv \varphi_{12}(\mathbf{x}_1) - \varphi_{12}(\mathbf{x}_2)$$

$$= \frac{1}{4\pi\xi}\, q(z) \int_{-\infty}^{\infty} dz' \{ \int_{C_1} dx_1'\, dy_1'\, \alpha_1(x_1',y_1') \frac{1}{R_{11}} - \int_{C_2} dx_2'\, dy_2'\, \alpha_2(x_2',y_2') \frac{1}{R_{12}} \}$$

$$- \frac{1}{4\pi\xi}\, q(z) \int_{-\infty}^{\infty} dz' \{ \int_{C_1} dx_1'\, dy_1'\, \alpha_1(x_1',y_1') \frac{1}{R_{21}} - \int_{C_2} dx_2'\, dy_2'\, \alpha_2(x_2',y_2') \frac{1}{R_{22}} \} \qquad (4.4.1)$$

which we obtained by assuming a separated form (4.1.2) for the charge density and by assuming a small Helmholtz parameter β. The $1/4\pi R$ factors here are in fact the 3D Poisson free-space propagators. This

propagator has the less glamorous name of being the electrostatic potential of a (1/ε)-size point charge (in "free space" of course), so by assuming the transmission line limit of small Helmholtz parameter β, we arrive at this electrostatics Poisson propagator appearing in the integrals. These propagators are "propagating" the effect of charges on the conductor surfaces to their destinations $\mathbf{x_1}$ and $\mathbf{x_2}$ in Fig 4.2 .

We then did the dz' integral over (-∞,∞) making use of integral (4.4.5),

$$\int_{-\infty}^{\infty} dz' \left(\frac{1}{R_{12}} - \frac{1}{R_{22}} \right) = \ln(s_{22}^2/s_{12}^2) \qquad (4.4.5)$$

and arrived at

$$V(z) = q(z) \frac{1}{4\pi\xi} \{ \int_{C_1} dx_1' dy_1' \alpha_1(x_1',y_1') \ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2' dy_2' \alpha_2(x_2',y_2') \ln(s_{22}^2/s_{12}^2) \} . \qquad (4.4.6)$$

This is really four terms and one recognizes $-\frac{1}{4\pi}\ln(R^2)$ in the form $-\frac{1}{4\pi}\ln(s_{ij}^2)$ as the 2D Poisson propagator just discussed above, and the s_{ij} are the 2D transverse distances shown in Fig 4.3. So here we see a very clear example of doing the 3D → 2D transition.

Case 3

Another transition example is the "scaling boundary condition" of Section 5 (b). We started there with

$$\varphi_t(\mathbf{x}) = \int_{-\infty}^{\infty} dz' \{ \int_{C_1} dx_1' dy_1' \alpha_1(x_1',y_1') \frac{1}{R_1} - \int_{C_2} dx_2' dy_2' \alpha_2(x_2',y_2') \frac{1}{R_2} \} \qquad (5.1.2)$$

and we moved the observation point **x** far away from the transmission line. The result in this limit was found to be

$$\varphi_t(\mathbf{x}) \approx \ln(s_2^2/s_1^2) \quad \text{// limiting form as point } \mathbf{x} = (x,y) \text{ moves far from the conductors} \qquad (5.3.11)$$

In this case, we had earlier done the following separation of the full potential

$$\varphi(x,y,z) = \frac{1}{4\pi\varepsilon} q(z) \varphi_t(x,y) \qquad (5.1.1)$$

so the limit shown for φ_t says

$$\varphi(x,y,z) \approx (q/\varepsilon) \frac{1}{4\pi} \ln(s_2^2/s_1^2) = (q/\varepsilon) \frac{1}{4\pi} \ln(s_2^2) - (q/\varepsilon) \frac{1}{4\pi} \ln(s_1^2) \qquad (J.3)$$

and we interpret this as being the sum of the 2D free-space propagations of charges ±q(z)dz to our distant point. We are so far from the transmission line that these charges appear as 2D point charges which form a little electric dipole as shown in Section 5.4 (b).

Case 4

As a third example, we consider a simple generic situation alluded to as our "abrupt" transition. Start again with

$$-(\nabla^2+k^2)\, f(\mathbf{x}) = h(\mathbf{x}) \quad \Rightarrow \quad f(\mathbf{x}) = \int d^3x' \, [e^{-jkR}/4\pi R]\, h(\mathbf{x}') + \text{homogeneous solutions}$$

The Helmholtz Equation — particular solution *(H.1.9)*

We changed the source name from s(x) to h(x) to avoid confusion with distance s below. We now assume that f(**x**) = f(x,y). What happens to the Helmholtz integral on the right?

$$f(\mathbf{x}) = \int d^3x' \, [e^{-jkR}/4\pi R]\, h(\mathbf{x}') = \int dx' \int dy' \int_{-\infty}^{\infty} dz' \; h(x',y') \, [e^{-jkR}/4\pi R]$$

where

$$s = \sqrt{(x-x')^2 + (y-y')^2} \quad \text{and} \quad R = \sqrt{s^2 + z'^2} \; .$$

Then

$$f(\mathbf{x}) = \int dx' \int dy' \, h(x',y') \, \frac{1}{4\pi} \int_{-\infty}^{\infty} dz' \, \frac{e^{-jkR}}{R} \qquad R = \sqrt{s^2 + z'^2} \; .$$

We can do the dz' integral as follows:

$$R^2 = s^2 + z'^2 \quad \Rightarrow \quad RdR = z'dz'$$

so

$$\int_{-\infty}^{\infty} dz' \, \frac{e^{-jkR}}{R} = \int_{-\infty}^{\infty} \frac{RdR}{z'} \, \frac{e^{-jkR}}{R} = \int_{-\infty}^{\infty} dR \, \frac{e^{-jkR}}{\sqrt{R^2-s^2}} = \int_{-\infty}^{\infty} dR \, \frac{\cos(kR)}{\sqrt{R^2-s^2}}$$

$$= 2 \int_{0}^{\infty} dR \, \frac{\cos(kR)}{\sqrt{R^2-s^2}} \; . \qquad \text{(J.4)}$$

We then take note of the following integral in GR7 3.754.2 page 435,

$$2. \qquad \int_0^{\infty} \frac{\cos(ax)\, dx}{\sqrt{\beta^2 + x^2}} = K_0(a\beta) \qquad [a > 0, \quad \mathrm{Re}\, \beta > 0]$$

which then says

$$\int_{-\infty}^{\infty} dR \, \frac{\cos(kR)}{\sqrt{R^2-s^2}} = K_0(k\sqrt{-s^2}\,) = K_0(-jks) \qquad z = -jks \qquad \text{phase}(z) = -\pi/2 \qquad \text{(J.5)}$$

$$(ze^{\pi j/2}) = zj = ks$$

But NIST p 250 says

10.27.8

$$K_\nu(z) = \begin{cases} \frac{1}{2}\pi i e^{\nu\pi i/2} H_\nu^{(1)}\left(ze^{\pi i/2}\right), & -\pi \le \operatorname{ph} z \le \frac{1}{2}\pi, \\ -\frac{1}{2}\pi i e^{-\nu\pi i/2} H_\nu^{(2)}\left(ze^{-\pi i/2}\right), & -\frac{1}{2}\pi \le \operatorname{ph} z \le \pi. \end{cases}$$

so that

$$K_0(-jks) = \pi(j/2)H_0^{(1)}(ks) \tag{J.6}$$

and then

$$\int_{-\infty}^{\infty} dz' \frac{e^{-jkR}}{R} = 2 \int_{-\infty}^{\infty} dR \frac{\cos(kR)}{\sqrt{R^2-s^2}} = j\pi H_0^{(1)}(ks)\,. \tag{J.7}$$

Finally

$$f(\mathbf{x}) = \int dx' \int dy'\, h(x',y') \frac{1}{4\pi} \int_{-\infty}^{\infty} dz' \frac{e^{-jkR}}{R} \qquad R = \sqrt{s^2 + z'^2}$$

$$= \int dx' \int dy'\, h(x',y') \frac{1}{4\pi}\, j\pi H_0^{(1)}(ks)$$

$$= \int dx' \int dy'\, h(x',y')[\,(j/4)\, H_0^{(1)}(ks)] \qquad s^2 (x-x')^2 + (y-y')^2$$

and once again we have transitioned from the 3D propagator $\frac{e^{-jkR}}{R}$ to the 2D one $(j/4)\, H_0^{(1)}(ks)$.

Case 5

In the k = 0 limit this becomes a transition from 3D propagator $\frac{1}{R}$ to 2D propagator $-\frac{1}{2\pi}\ln(s)$ as follows :

$$f(\mathbf{x}) = \int d^3x' \,[1/4\pi R]\, h(\mathbf{x}') = (1/4\pi) \int dx' \int dy'\, h(x',y') \int_{-\infty}^{\infty} \frac{dz'}{\sqrt{s^2 + z'^2}}\,.$$

But now the dz' integral is logarithmically divergent so we install a very large cutoff Λ and write

$$\int_{-\infty}^{\infty} \frac{dz'}{\sqrt{s^2 + z'^2}} \rightarrow \int_{-\Lambda/2}^{\Lambda/2} \frac{dz'}{\sqrt{s^2 + z'^2}} = 2 \int_{0}^{\Lambda/2} \frac{dz'}{\sqrt{s^2 + z'^2}} = 2 \ln[\, z' + \sqrt{z'^2 + s^2}\,]\,|^{\Lambda/2}_0$$

$$= 2\ln[\Lambda/2 + \sqrt{(\Lambda/2)^2 + s^2}\,] - 2 \ln s$$

$$\approx 2\ln(\Lambda) - 2\ln s = -2\ln(s/\Lambda)\ . \qquad \text{(J.8)}$$

Now we apply the argument above about ignoring constants to get,

$$\int_{-\infty}^{\infty} \frac{dz'}{\sqrt{s^2 + z'^2}} = -2\ln s \qquad \text{// ignoring constants}$$

$$f(\mathbf{x}) = \int dx' \int dy'\, h(x',y')\,(1/4\pi)\,(-2\ln s) = \int dx' \int dy'\, h(x',y')\,[-\frac{1}{2\pi}\ln(s)]$$

and so we have transitioned in this case from the 3D Poisson propagator to the 2D one.