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Appendix K : The Network Model: Comparison of Network and Maxwell Views

(a) The Network Model

The usual network model of a 2-conductor transmission line is an infinite repetition of differentially small R,L,C,G segments as shown here between the vertical red lines,

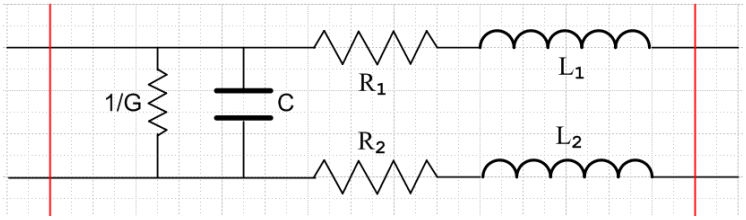


Fig K.1

This is of course electrically identical to the following,

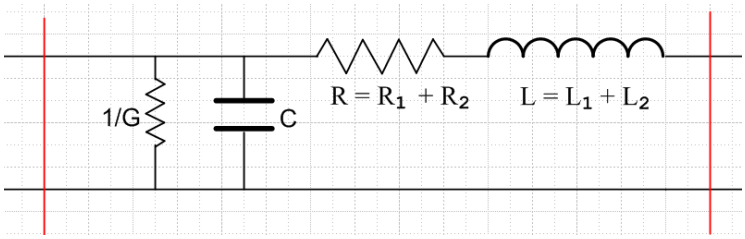


Fig K.2

where $R = R_1 + R_2$ and $L = L_1 + L_2$. In the circuit diagrams, it is implied that R,L,C,G are all quantities *per unit length* of the transmission line. Thus, if the distance between the two red lines is δ, the values of the lumped parameters in Fig K.2 are Rδ,Lδ,Cδ,Gδ. For example, if δ doubles, the total conductance of the segment doubles since it is a measure of current flowing between the conductors. The model implied by the picture is then the limit as $\delta \to 0$.

We wish to compare this "network model" to our Maxwell equation results. To start, we note that the impedance of a capacitor C and inductor L operating at frequency ω is determined by

$$Q = CV \quad \Rightarrow \quad I = \partial_t Q = C\,\partial_t V \quad \Rightarrow \quad I = j\omega CV \quad \Rightarrow \quad Z_C = V/I = 1/(j\omega C)$$

$$V = L\,\partial_t I \quad \Rightarrow \quad V = j\omega LI \quad \Rightarrow \quad Z_L = V/I = j\omega L \ . \qquad (K.1)$$

Note that the "admittance" of a capacitor is $Y_C = 1/Z_C = j\omega C$. We can then combine the G and C elements together into a single element having $\mathbf{y} = G+j\omega C$, since parallel admittances are additive. Similarly, we combine the two series elements R and L into impedance $\mathbf{z} = R+j\omega L$. The network picture is then,

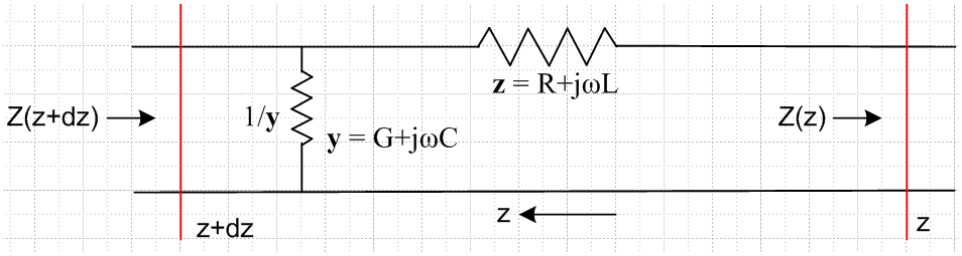


Fig K.3

We use King's bolded symbols **y** and **z** and of course **z** is unrelated to distance z. Here we arbitrarily have the z axis pointing to the *left* (!), and the vertical red lines area placed at z and z+dz so that $\delta = dz$. The impedance looking into the transmission line from the left is Z(z+dz) at z+dz and is Z(z) at z.

(b) Network Model Characteristic Impedance

Since the impedance $1/\mathbf{y}$ is in parallel with the impedance $\mathbf{z} + Z(z)$ we have

$$Z(z+dz) = (\mathbf{y}dz)^{-1} \parallel (\mathbf{z}dz + Z(z)) = \text{product over sum} = \frac{(\mathbf{y}dz)^{-1}\,(\mathbf{z}dz + Z(z))}{(\mathbf{y}dz)^{-1} + (\mathbf{z}dz + Z(z))}$$

$$= \frac{(\mathbf{z}dz + Z(z))}{1 + (\mathbf{y}dz)\,(\mathbf{z}dz + Z(z))} \approx [Z(z) + \mathbf{z}dz]\,[1 - (\mathbf{y}dz)(\mathbf{z}\,dz + Z(z)]$$

$$\approx [Z(z) + \mathbf{z}dz]\,[1 - (\mathbf{y}dz)\,Z(z)] \qquad \text{// dropping order } (dz)^2$$

$$\approx Z(z) + [\mathbf{z} - \mathbf{y}Z^2(z)]\,dz\ . \qquad \text{// dropping order } (dz)^2 \text{ again}$$

Therefore Z(z) must solve this non-linear first order differential equation,

$$\frac{dZ(z)}{dz} = \mathbf{z} - \mathbf{y}Z^2(z) \qquad \text{or} \qquad \frac{dZ(z)}{dz} + \mathbf{y}\,Z^2(z) = \mathbf{z}\ . \tag{K.2}$$

The most general solution to this equation is

$$Z(z) = \sqrt{\frac{\mathbf{z}}{\mathbf{y}}}\ \text{th}\,(\sqrt{\mathbf{zy}}\ z + C) \qquad C = \text{constant} \tag{K.3}$$

since

$$\partial_{\mathbf{z}}Z = \sqrt{\frac{\mathbf{z}}{\mathbf{y}}} * \sqrt{\mathbf{zy}}\ \text{sech}^2(\sqrt{\frac{\mathbf{z}}{\mathbf{y}}}\ z + C) = \mathbf{z}\ [\,1 - \text{th}^2(\sqrt{\frac{\mathbf{z}}{\mathbf{y}}}\ z + C)\,]$$

$$= \mathbf{z}\ [\ 1 - \frac{\mathbf{y}}{\mathbf{z}} Z^2(z)] = \mathbf{z} - \mathbf{y}\, Z^2(z)\ .$$

If the transmission line is of finite length running from z = L (left end) to z =0 (right end), and if the line is *terminated* at z = 0 by some impedance Z_t, we must have $Z(0) = Z_t$ so that

$$Z_t = Z(0) = \sqrt{\frac{\mathbf{z}}{\mathbf{y}}}\ \text{th}\,(\sqrt{\mathbf{zy}}\ 0\ + C\ ) = \sqrt{\frac{\mathbf{z}}{\mathbf{y}}}\ \text{th}(C)$$

$$\Rightarrow \qquad C = \text{th}^{-1}(\sqrt{\frac{\mathbf{y}}{\mathbf{z}}}\ Z_t)$$

so then the solution is

$$Z(z) = \sqrt{\frac{\mathbf{z}}{\mathbf{y}}}\ \text{th}\,[\ \sqrt{\mathbf{zy}}\ \ z\ + \text{th}^{-1}(\sqrt{\frac{\mathbf{y}}{\mathbf{z}}}\ Z_t)\ ]$$

and at the left end we find

$$Z(L) = \sqrt{\frac{\mathbf{z}}{\mathbf{y}}}\ \text{th}\,[\ \sqrt{\mathbf{zy}}\ \ L\ + \text{th}^{-1}(\sqrt{\frac{\mathbf{y}}{\mathbf{z}}}\ Z_t)\ ]\ .$$

If we take $L \to \infty$ (line becomes infinitely long) , then th [...] $\to 1$ and we find

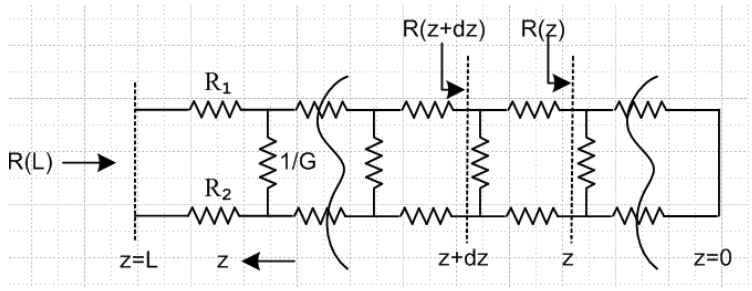
$$Z(\infty) = \sqrt{\frac{\mathbf{z}}{\mathbf{y}}}\ ,$$

so the impedance looking into the left end of the *infinite* transmission line is independent of the termination value Z_t at z = 0. This infinite line impedance is called the **characteristic impedance** Z_0 and we have shown then that

$$Z_0 = \sqrt{\frac{\mathbf{z}}{\mathbf{y}}}\ = \sqrt{\frac{R+j\omega L}{G+j\omega C}}\ . \qquad \text{(K.4)}$$

Since this is the same result obtained from Maxwell's equations in (4.11.16), one is motivated to regard the network transmission line model as a *correct* model, and then the network model parameters R,L,G,C can be identified with the parameters obtained from Maxwell's equations.

Reader Exercise: Consider this purely resistive finite ladder network shorted at the right end,



(1) Using the results above, show that

$$R(L) = \sqrt{R_3/G}\ \tanh\left(\sqrt{R_3 G}\ L\right) \qquad \text{where } R_3 \equiv R_1 + R_2 .$$

(2) Show that

$$R(L) \approx \sqrt{R_3/G} \qquad \text{if } L \gg 1/\sqrt{R_3 G} .$$

Thus, for large L the fact that the line is shorted at the right end makes no difference.

(3) Show that for finite L :

$R(L) \to R_3 L$ as $G \to 0$ no conductance
$R(L) \to 0$ as $R_3 \to 0$ no wire resistance

Both limits should seem obvious.

(c) Network Model Transmission Line Equations

We now switch the z axis back to its usual direction (increasing to the right), and we label currents and voltages on our transmission line section,

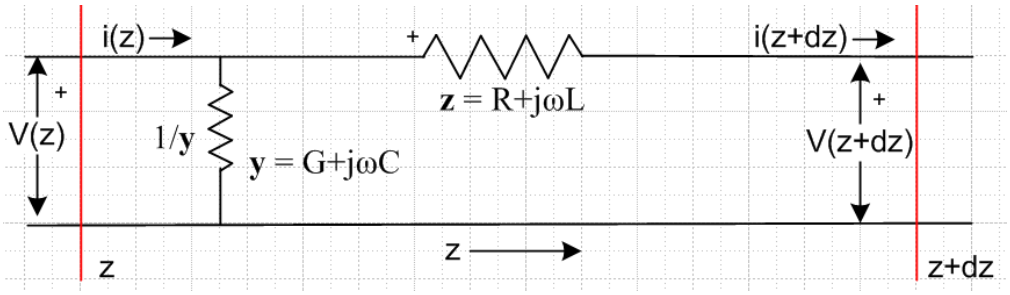


Fig K.4

Staring at the picture, it seems clear that

$$i(z) - i(z+dz) = \text{current going down through impedance } 1/(\mathbf{y}dz) = \frac{V(z)}{1/(\mathbf{y}dz)} = \mathbf{y}dz\, V(z)$$

and therefore

$$-\frac{di(z)}{dz} = \mathbf{y}\, V(z) \ .$$

Meanwhile, the voltage across the impedance **z** is V(z) - V(z+dz) so

$$V(z) - V(z+dz) = i(z)\, \mathbf{z}$$

and therefore

$$-\frac{dV(z)}{dz} = \mathbf{z}\, i(z) \ .$$

Thus we have shown that

$$\frac{dV(z)}{dz} = -\,\mathbf{z}\, i(z) \qquad \frac{di(z)}{dz} = -\,\mathbf{y}\, V(z)$$

with

$$\mathbf{z} = R + j\omega L \qquad \mathbf{y} = G + j\omega C \ . \qquad \text{(K.5)}$$

Differentiating these equations with respect to z, we find that

$$\frac{d^2V(z)}{dz^2} - \mathbf{zy}\, V(z) = 0 \qquad \frac{d^2i(z)}{dz^2} - \mathbf{zy}\, i(z) = 0 \qquad \text{(K.6)}$$

But (K.5) and (K.6) are the same transmission line equations obtained from Maxwell's equations as shown in (4.11.14b) and (4.11.15). Thus we are further encouraged in our use of the network model to represent a transmission line. Since the equations found from Maxwell's equations were qualified as being questionable at very low frequencies, the network model is also suspect at very low ω

(d) Network Model Parameters obtained from Maxwell's Equations

The main results of Chapter 4 appear in summary box (4.11.34) from which we quote in part,

$$\frac{dV(z)}{dz} = -\,\mathbf{z}\, i(z) \qquad \left(\frac{d^2}{dz^2} - \mathbf{zy}\right) V(z) = 0 \qquad \mathbf{z} = R + j\omega L \qquad \text{transmission line equations}$$

$$\frac{di(z)}{dz} = -\,\mathbf{y} V(z) \qquad \left(\frac{d^2}{dz^2} - \mathbf{zy}\right) i(z) = 0 \qquad \mathbf{y} = G + j\omega C \qquad \textit{(4.11.14), (4.11.15)}$$

$$\mathbf{z} = Z_{s1} + Z_{s2} + j\omega L_e \quad \textit{(4.11.17)} \qquad X_L \equiv \omega L_e \ , \ X_C \equiv 1/(\omega C)$$

$$\mathbf{y} = j\omega C' = j\omega C + (\sigma_d/\varepsilon_d)C \quad \textit{(4.11.24)} \qquad G = (\sigma_d/\varepsilon_d)C \quad \textit{(4.11.25)}$$

$$R = \mathrm{Re}(Z_{s1} + Z_{s2})$$

$$L = L_e + (1/\omega)\, \mathrm{Im}(Z_{s1} + Z_{s2})$$

$L_e = (\mu_d/4\pi)K$ *(4.11.29) and (4.11.30)*
$C = 4\pi\varepsilon_d/K$ *(4.11.26)*
$G = 4\pi\sigma_d/K$ *(4.11.26) + (4.11.25)* (K.7)

Thus, we make the connection between the network parameters and the Maxwell calculation parameters as follows:

$R = \mathrm{Re}(Z_{s1} + Z_{s2})$
$L = L_e + (1/\omega)\,\mathrm{Im}(Z_{s1} + Z_{s2})$
$L_e = (\mu_d/4\pi)K$
$G = 4\pi\sigma_d/K$
$C = 4\pi\varepsilon_d/K$ (K.8)

where K is the dimensionless real integral in Chapter 4, see (4.4.8). Recall that this integral requires knowledge of both the conductor geometry as well as the normalized transverse surface charge distributions on the conductors. Here ε, μ and σ are for the dielectric between the conductors.

(e) Low frequency case (no skin effect)

At low frequencies, when conductors are not extremely close together, the current densities are close to uniform (see for example Fig 6.16), so that J_z = I/area for each conductor. This uniformity is exact for a conductor which is the central conductor of a coaxial cable, as studied in Chapter 2. There we found at low frequency that

$$Z_{s1}(\omega) = \frac{1}{\sigma_1 \pi a^2} + j\omega \frac{\mu_1}{8\pi} \qquad \text{// low frequency limit} \qquad \textit{(2.4.12)}$$

$$\Rightarrow \qquad \mathrm{Re}(Z_{s1}) = \frac{1}{\sigma_1 \pi a^2} \qquad \text{and} \qquad \mathrm{Im}(Z_{s1}) = \omega \frac{\mu_1}{8\pi} \ . \qquad \text{(K.9)}$$

From (K.8) we then find that for low frequencies and parallel round conductors,

$$R = \frac{1}{\sigma_1 \pi a_1^2} + \frac{1}{\sigma_2 \pi a_2^2} = R_{dc1} + R_{dc2} \qquad \text{(K.10)}$$

$$L = L_e + \left(\frac{\mu_1}{8\pi} + \frac{\mu_2}{8\pi}\right) = L_e + (L_{i1} + L_{i2}) \ . \qquad \text{(K.11)}$$

In this case parameter R is just the sum of the DC resistances of the conductors (per unit length), and parameter L is the sum of the external inductance L_e and the internal inductances of the two wires. Here σ_i and μ_i are for the material from which conductor C_i is constructed. The external inductance L_e can be interpreted as the inductance of the red wire loop below,

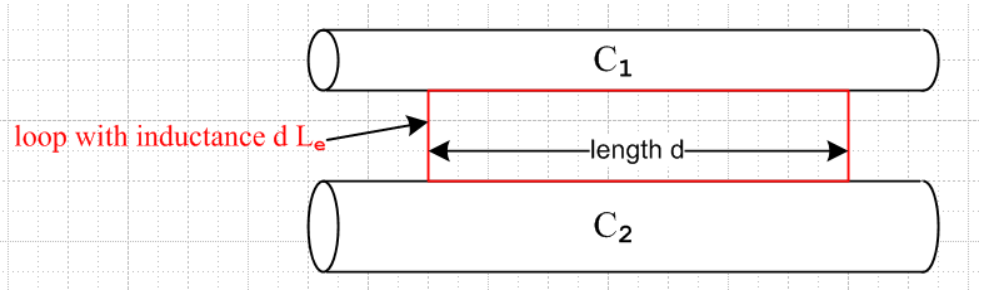


Fig K.3

The sides of the red loop make contact on any line on the conductor surfaces, though here we show it having its minimal size. The red loop may in fact be replaced by any loop, possibly non-planar, which captures all the external magnetic flux passing between the conductors. See Fig 4.11 and discussion there.

Note that L_e is *not* the inductance of a rectangular thin wire loop *in isolation* occupying the red outline above, but rather $L_e = (\mu_d/4\pi)K$ as in (K.8) above, where K is related to the capacitance between the conductors. If both conductors are round and very thin and separated by distance b, we know from (4.5.7) that $K = 4\ln(b/\sqrt{a_1 a_2})$ and then $L_e = (\mu_d/\pi)\ln(b/\sqrt{a_1 a_2})$.

Although we have not formally proven it, it seems clear that for arbitrary conductor cross sections (not too closely spaced) the following equations will apply at low frequency :

$$R = \frac{1}{\sigma_1 A_1} + \frac{1}{\sigma_2 A_2} \qquad A_i = \text{cross section area of } C_i \qquad \text{(K.12)}$$

$$L = L_e + (L_{i1} + L_{i2}) \ . \qquad \text{(K.13)}$$

Appendix C computes the DC L_i for various conductor cross section shapes. One result quoted there from the literature is that for a square conductor,

$$L_i = (\mu_i/8\pi)\ [0.96639] \ . \qquad \textit{(C.4.12)}$$

Thus the L_i for a square cross-section conductor is barely different from that of a round conductor.

(f) High frequency case for round conductor (strong skin effect)

At high frequencies there is a pronounced skin effect. In Chapter 2 for a round conductor C_1 at high frequency (and with a symmetric current distribution) we found that

$$Z_{s1}(\omega) \approx \frac{1}{\sigma_1(2\pi a_1)\delta_1}\ (1+j) \qquad \delta_1 << 4a_1 \qquad \textit{(2.4.16)}$$

$$\mathrm{Re}(Z_{s1}) = \mathrm{Im}(Z_{s1}) = \frac{1}{\sigma_1(2\pi a_1)\delta_1}$$

where $\delta_1 = \sqrt{2/(\omega\mu_1\sigma_1)}$ is the skin depth and a_1 the wire radius.

From (K.8) we find that for high frequencies and round conductors,

$$R = \frac{1}{\sigma_1(2\pi a_1)\delta_1} + \frac{1}{\sigma_2(2\pi a_2)\delta_2}$$

$$L = L_e + (1/\omega)\,\mathrm{Im}(Z_{s1} + Z_{s2}) = L_e + (1/\omega)\,R \quad . \tag{K.14}$$

In this case, we recognize $2\pi a_1\delta_1$ as the effective current carrying cross-sectional area of round conductor C_1 (the area of the current sheath), so the expression for R is quite intuitive. Since,

$$\delta \equiv \sqrt{2/\omega\mu\sigma} \qquad \Rightarrow \qquad 1/\delta_1 = \sqrt{(\omega\mu_1\sigma_1)/2} \qquad \text{and} \qquad 1/\omega = \mu_1\sigma_1\delta_1^2/2 \tag{K.15}$$

we may write

$$L_i(\omega) = (1/\omega)\,\mathrm{Im}(Z_s) = (1/\omega)\frac{1}{\sigma_1(2\pi a_1)\delta_1} = (1/\omega)\frac{1}{\sigma_1(2\pi a_1)}\sqrt{(\omega\mu_1\sigma_1)/2} = \frac{1}{2\pi a_1}\sqrt{\frac{\mu_1}{2\sigma_1\omega}} \tag{K.16}$$

so $L_i(\omega) \sim 1/\sqrt{\omega}$. Expressing L_i instead in terms of δ_1 we find

$$L_i(\delta_1) = (1/\omega)\,\mathrm{Im}(Z_s) = (1/\omega)\frac{1}{\sigma_1(2\pi a_1)\delta_1} = \mu_1\sigma_1(\delta_1^2/2)\frac{1}{\sigma_1(2\pi a_1)\delta_1} = \mu_1\,(1/4\pi)\,(\delta_1/a_1)$$

$$= \frac{\mu_i}{8\pi}\,[\,2\,(\delta_1/a_1)\,] \quad . \tag{K.17}$$

The DC internal inductance of a thin shell of radius a and thickness d is shown in Appendix C.6 to be

$$L_i = \mu_i\,\frac{1}{6\pi}\,(d/a) = \frac{\mu_i}{8\pi}\,[\,(4/3)(d/a)\,] \qquad \text{thin shell, valid for } d << a \tag{C.6.8}$$

so the high frequency internal inductance of a round wire is the same as the DC internal inductance a shell of thickness d = (3/2)δ which seems fairly reasonable. The above expression (C.6.8) shows that the inductance of a thin cylindrical shell is linear in the shell thickness d, so we expect that the high frequency L_i of a round wire should be linear in δ, and thus proportional to $1/\sqrt{\omega}$.

Section 2.5 shows how to handle non-round conductors and non-symmetric current distributions by replacing 2πa by an effective active perimeter D. Chapter 4.11 (b) formalizes this notion, giving the effective perimeter in (4.11.10).

In Section D.10 and D.11 the claims made in the last two sections regarding surface impedance are vindicated when one uses the surface impedance averaged over the round wire surface, see (D.10.17) for large ω and (D.11.10) for low ω.