This is my original writeup of this appendix, it is now all installed.

Appendix K : Line Parameters: Comparison of Network and Maxwell Views 1
(a) The Network Model..... 1
(b) Characteristic Impedance in the Network Model..... 1
(c) Characteristic impedance computed from Maxwell's Equations..... 3
(d) Low frequency case (no skin effect) 4
(e) High frequency case for round conductor (strong skin effect) 5

Appendix K : Line Parameters: Comparison of Network and Maxwell Views

(a) The Network Model

The usual model of a transmission line is an infinite repetition of differentially small R,L,C,G segments as shown here between the vertical red lines,

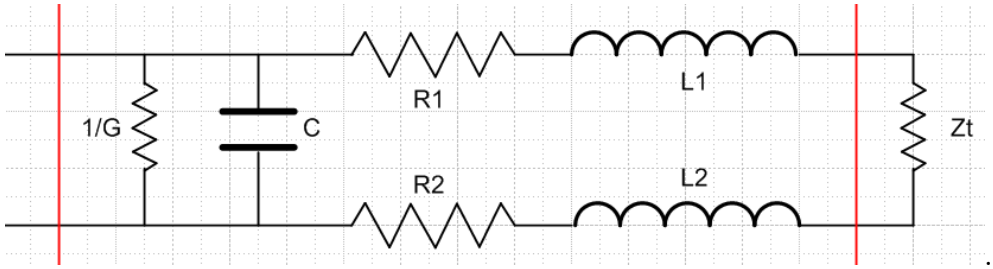


Fig K.1

This is of course electrically identical to the following

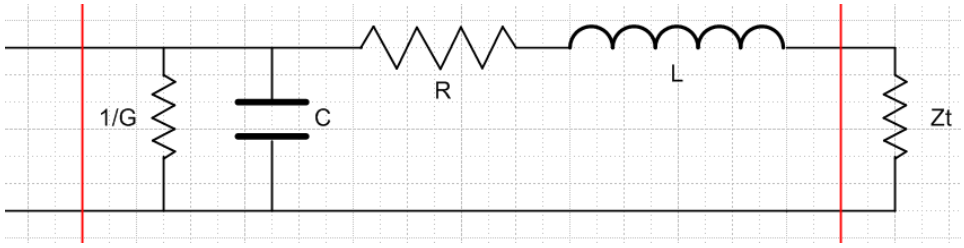


Fig K.2

where $R = R_1 + R_2$ and $L = L_1 + L_2$. In the circuit diagrams, it is implied that R,L,C,G are all quantities *per unit length* of the transmission line. Thus, if the distance between the two red lines is δ, the values of the lumped parameters in Fig K.2 are Rδ,Lδ,Cδ,Gδ. For example, if δ doubles, the total conductance of the segment doubles since it is a measure of current flowing between the conductors. The model implied by the picture is then the limit as $\delta \to 0$.

We wish to compare these parameters R,L,C,G with our parameters calculated from Maxwell's equations.

(b) Characteristic Impedance in the Network Model

The first step to this end is to compute the impedance Z_{in} one would see looking into the left end of the above "two-port network" consisting of a single segment terminated by some impedance Z_t. If we set this computed impedance Z_{in} equal to Z_t, the resulting Z_t must then be the "characteristic impedance" of the infinite transmission line looking in from the left end.

The impedance of a capacitor C and inductor L operating at frequency ω is determined by

$$Q = CV \quad \Rightarrow \quad I = \partial_t Q = C\partial_t V \quad \Rightarrow \quad I = j\omega CV \quad \Rightarrow \quad Z_C = V/I = 1/(j\omega C)$$

$$V = L\partial_t I \quad \Rightarrow \quad V = j\omega LI \quad \Rightarrow \quad Z_L = V/I = j\omega L \ . \qquad \text{(K.1)}$$

If we momentarily set $\delta = 1$, then using the usual circuit rules for computing impedance we see that

$$Z_{in} = (1/G) \ || \ Z_C \ || \ (R+Z_L+ Z_t) \qquad Z_C = 1/(j\omega C) \qquad Z_L = j\omega L \ .$$

since these three elements are in parallel. For parallel elements, since all have the same voltage V_{in}, it is easier to add up the inverse impedances ("admittances"), which basically adds up the three currents to get the total current (a fact easily shown) so then

$$Z_{in}^{-1} = G + Z_C^{-1} + (R + Z_L + Z_t)^{-1}$$

or

$$Z_{in}^{-1} = G + j\omega C + (R + j\omega L + Z_t)^{-1} \ . \qquad \text{(K.2)}$$

To find the characteristic impedance we set $Z_{in} = Z_t = Z_0$ to get

$$Z_0^{-1} = G + j\omega C + (R + j\omega L + Z_0)^{-1} \ . \qquad \text{(K.3)}$$

Now we reinstall our δ factors to get

$$Z_0^{-1} = (G+j\omega C)\delta + \frac{1}{(R+j\omega L)\delta + Z_0} \qquad \text{(K.4)}$$

or

$$[(R+j\omega L)\delta + Z_0] \, Z_0^{-1} = (G+j\omega C)\delta \, [(R+j\omega L)\delta + Z_0]$$

or

$$[(R+j\omega L)\delta + Z_0] = (G+j\omega C)\delta \, [(R+j\omega L)\delta Z_0 + Z_0^2] \ . \qquad \text{(K.5)}$$

We enter into Maple the equation in the form (K.4). The two quadratic solutions go into array elements s[1] and s[2], and s[1] gives the physical solution. This result includes terms of order δ^0, δ and δ^2 and we then take the limit $\delta \to 0$ simply by setting $\delta = 0$. The extra Maple commands are just guides to help Maple obtain the result in a simple form.

```
eq := 1/Z0 = delta*G + j*omega*delta*C + 1/(delta*R+j*omega*delta*L+Z0);
```

$$eq := \frac{1}{Z0} = \delta G + j\omega\delta C + \frac{1}{\delta R + j\omega\delta L + Z0}$$

```
s := solve(eq,Z0):
z0 := factor(s[1]);
```

$$z0 := -\frac{1}{2}\frac{G\delta R + jG\omega\delta L + j\omega C\delta R - \omega^2 C\delta L + \sqrt{-(\delta^2 GR + j\delta^2 G\omega L + j\delta^2\omega CR - \delta^2\omega^2 CL + 4)(jG - C\omega)(jR - \omega L)}}{G + j\omega C}$$

```
t := subs(delta=0,z0): Z0 := sqrt(t^2);
```

$$Z0 := \sqrt{\frac{j\omega L + R}{G + j\omega C}}$$

Thus we arrive at the famous result for the characteristic impedance of a transmission line in the network model described above,

$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} \quad . \tag{K.6}$$

(c) Characteristic impedance computed from Maxwell's Equations

The main results of Chapter 4 appear in summary box (4.11.30) from which we quote in part,

$\frac{\partial V(z)}{\partial z} = -\mathbf{z}\, i(z)$ where $\mathbf{z} = R + j\omega L$ transmission line equations

$\frac{\partial i(z)}{\partial z} = -\mathbf{y} V(z)$ *(4.11.11)* $\mathbf{y} = G + j\omega C$ *(4.11.12)*

$\mathbf{z} = Z_{s1} + Z_{s2} + j\omega L_e$ *(4.11.13)* $X_L \equiv \omega L_e$, $X_C \equiv 1/(\omega C)$

$\mathbf{y} = j\omega C' = j\omega C + (\sigma/\varepsilon)C$ *(4.11.20)* $G = (\sigma/\varepsilon)C$ *(4.11.21)*

$R = \mathrm{Re}(Z_{s1} + Z_{s2})$

$L = L_e + (1/\omega)\,\mathrm{Im}(Z_{s1} + Z_{s2})$

$L_e = (\mu/4\pi)K$ *(4.11.25) and (4.11.26)*

$C = 4\pi\varepsilon/K$ *(4.11.22)*

$G = 4\pi\sigma/K$ *(4.11.22) + (4.11.21)* (K.7)

Assuming the transmission line carries a wave of the usual form $e^{j(kz-\omega t)}$, we know that

$V(z) = V(0)e^{-jkz} \quad \Rightarrow \quad \partial_z V(z) = jkV(z)$

$i(z) = i(0)e^{-jkz} \quad \Rightarrow \quad \partial_z i(z) = jki(z)$. (K.8)

Thus the transmission line equations above may be written

$jkV = -\mathbf{z}\, i$

$jk\, i = -\mathbf{y} V$. (K.9)

Multiply the first by i and the second by V and subtract the resulting equations to get

$$0 = -\mathbf{z}\, i^2 + \mathbf{y} V^2 \quad \Rightarrow \quad \mathbf{z}\, i^2 = \mathbf{y} V^2 \quad \Rightarrow \quad (V/i)^2 = \mathbf{z}/\mathbf{y} \tag{K.10}$$

Therefore we find that the characteristic impedance of the infinite transmission line is

$$Z_0 = V/i = \sqrt{\mathbf{z}/\mathbf{y}} = \sqrt{\frac{R+j\omega L}{G+j\omega C}} \tag{K.11}$$

where R,L,G,C are the parameters computed from Maxwell's equations. Since this result agrees exactly with our network computation, we conclude that these R,L,G,C parameters are the same in both the Maxwell and network models. Thus, we make the connection between the network parameters and the Maxwell calculation parameters as follows:

$$\begin{aligned} R &= \mathrm{Re}(Z_{s1} + Z_{s2}) \\ L &= L_e + (1/\omega)\, \mathrm{Im}(Z_{s1} + Z_{s2}) \\ L_e &= (\mu/4\pi)K \\ G &= 4\pi\sigma/K \\ C &= 4\pi\varepsilon/K \end{aligned} \tag{K.12}$$

where K is the dimensionless real integral in Chapter 4, see (4.4.8). Recall that this integral requires knowledge of both the conductor geometry as well as the normalized J_z current distributions inside the conductors. Here ε, μ and σ are for the dielectric between the conductors.

(d) Low frequency case (no skin effect)

At low frequencies where there is no skin effect, the conductor current densities are uniform, so that J_z = I/area for each conductor. In Chapter 2 for a round conductor C_1 at low frequency we found that

$$Z_s(\omega) = \frac{1}{\sigma_1 \pi a^2} + j\omega \frac{\mu_1}{8\pi} \qquad \text{// low frequency limit} \qquad \textit{(2.4.12)}$$

$$\Rightarrow \qquad \mathrm{Re}(Z_s) = \frac{1}{\sigma_1 \pi a^2} \qquad \text{and} \qquad \mathrm{Im}(Z_s) = \omega \frac{\mu_1}{8\pi} \ . \tag{K.13}$$

From (K.12) we then find that for low frequencies and round conductors,

$$R = \frac{1}{\sigma_1 \pi a_1^2} + \frac{1}{\sigma_2 \pi a_2^2} = R_{dc1} + R_{dc2} \tag{K.14}$$

$$L = L_e + \left(\frac{\mu_1}{8\pi} + \frac{\mu_2}{8\pi} \right) = L_e + (L_{i1} + L_{i2}) \ . \tag{K.15}$$

In this case parameter R is just the sum of the DC resistances of the conductors (per unit length), and parameter L is the sum of the external inductance L_e and the internal inductances of the two wires. Here

σ_i and μ_i are for the material from which conductor C_i is constructed. The external inductance L_e can be interpreted as the inductance of the red wire loop below,

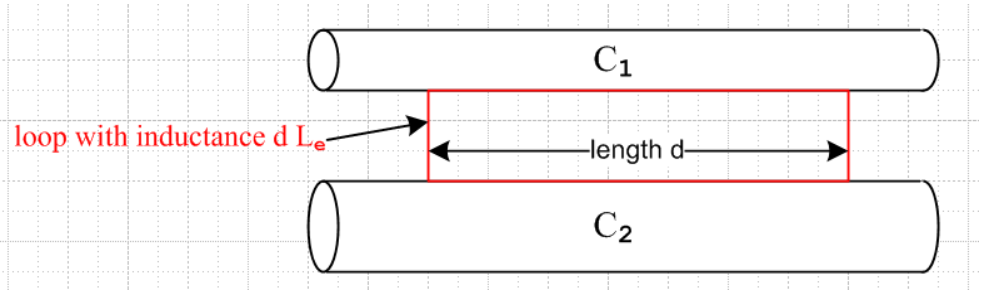


Fig K.3

The sides of the red loop make contract on any line on the conductor surfaces, though here we show it having its minimal size. The red loop may in fact be replaced by any loop, possibly non-planar, which captures all the external magnetic flux passing between the conductors. See Fig 4.11 and discussion there.

Although we have not formally proven it, it seems clear that for arbitrary conductor cross sections the following equations will apply at low frequency :

$$R = \frac{1}{\sigma_1 A_1} + \frac{1}{\sigma_2 \pi A_2} \qquad A_i = \text{cross section area of } C_i \qquad \text{(K.16)}$$

$$L = L_e + (L_{i1} + L_{i2}) \ . \qquad \text{(K.17)}$$

Appendix C computes the DC L_i for various conductor cross section shapes. One result quoted there from the literature is that for a square conductor,

$$L_i = (\mu_i/8\pi)\ [0.96639] \ . \qquad \textit{(C.4.11)}$$

Thus the L_i for a square cross-section conductor is barely different from that of a round conductor.

(e) High frequency case for round conductor (strong skin effect)

At high frequencies there is a pronounced skin effect. In Chapter 2 for a round conductor C_1 at high frequency (and with a symmetric current distribution) we found that

$$Z_s(\omega) \approx \frac{1}{\sigma_1 (2\pi a_1)\delta_1}\ (1+j) \qquad \delta_1 << 16a_1 \qquad \textit{(2.4.16)}$$

$$\mathrm{Re}(Z_s) = \mathrm{Im}(Z_s) = \frac{1}{\sigma_1 (2\pi a_1)\delta_1}$$

where here δ_1 is the skin depth $\delta_1 = \sqrt{2/(\omega\mu_1\sigma_1)}$ and a_1 is the wire radius.

From (K.12) we find that for high frequencies and round conductors,

$$R = \frac{1}{\sigma_1(2\pi a_1)\delta_1} + \frac{1}{\sigma_2(2\pi a_2)\delta_2}$$

$$L = L_e + (1/\omega)\,\mathrm{Im}(Z_{s1} + Z_{s2}) = L_e + (1/\omega)\,R \quad . \tag{K.18}$$

In this case, we recognize $2\pi a_1\delta_1$ as the effective current carrying cross-sectional area of round conductor C_1, so the expression for R is quite intuitive. Since,

$$\delta \equiv \sqrt{2/\omega\mu\sigma} \qquad \Rightarrow \qquad 1/\delta = \sqrt{(\omega\mu_1\sigma_1)/2} \qquad \text{and} \qquad 1/\omega = \mu_1\sigma_1\delta_1^2/2 \tag{K.19}$$

we may write

$$L_i(\omega) = (1/\omega)\,\mathrm{Im}(Z_s) = (1/\omega)\frac{1}{\sigma_1(2\pi a_1)\delta_1} = (1/\omega)\frac{1}{\sigma_1(2\pi a_1)}\sqrt{(\omega\mu_1\sigma_1)/2} = \frac{1}{2\pi a_1}\sqrt{\frac{\mu_1}{2\sigma_1\omega}} \tag{K.20}$$

so $L_i(\omega) \sim 1/\sqrt{\omega}$. Expressing L_i instead in terms of δ_1 we find

$$L_i(\delta_1) = (1/\omega)\,\mathrm{Im}(Z_s) = (1/\omega)\frac{1}{\sigma_1(2\pi a_1)\delta_1} = \mu_1\sigma_1(\delta_1^2/2)\frac{1}{\sigma_1(2\pi a_1)\delta_1} = \mu_1\,(1/4\pi)\,(\delta_1/a_1)$$

$$= \frac{\mu_i}{8\pi}\,[\,2\,(\delta_1/a_1)\,] \quad . \tag{K.21}$$

The DC internal inductance of a thin shell of radius a and thickness d is shown in Appendix C.6 to be

$$L_i = \mu_i\,\frac{1}{6\pi}\,(d/a) = \frac{\mu_i}{8\pi}\,[\,(4/3)(d/a)\,] \qquad \text{thin shell, valid for } d << a \tag{C.6.8}$$

so the high frequency internal inductance of a round wire is the same as the DC internal inductance a shell of thickness $d = (3/2)\delta$. This is perhaps reasonable since for the DC shell J_z is uniform over distance d, but for the skin effect case J_z decays down to 1/e in distance δ.

On the other hand, for the DC current interpretation above, this 1/e drop-off is offset by the fact that some current still exists inside the skin depth δ.

In any event, the above expression (C.6.8) shows that the inductance of a thin cylindrical shell is linear in the shell thickness d, so we expect that the high frequency L_i of a round wire should be linear in δ, and thus proportional to $1/\sqrt{\omega}$.

Section 2.5 shows how to handle non-round conductors and non-symmetric current distributions by replacing $2\pi a$ by an effective active perimeter D. For the special case of Chapter 6, everything can be computed exactly.