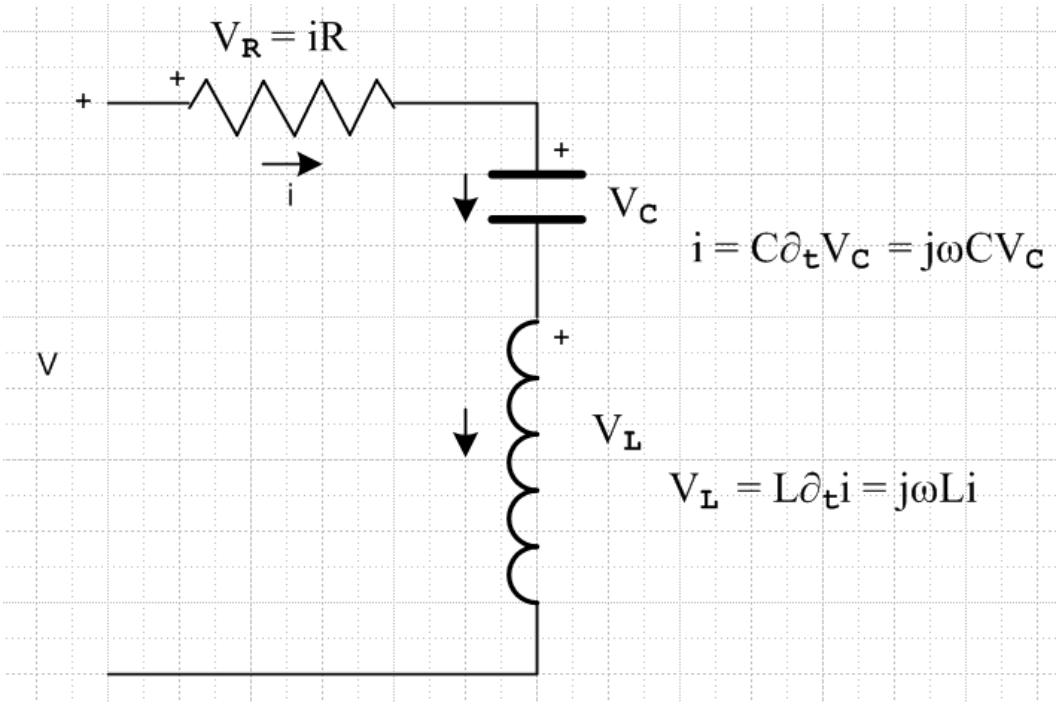
Reactance PhL 10.18.13

Comments 12/5/13. This is useful stuff. I got things cleared up, and then I treated the transmission line using this model (see below) and got some useful results. That might go into an appendix some time. King does this approach first in his book. This is really Chapter 4 related stuff and I have not gotten to Chapter 4 really even at this late date!

I am messing this up, so let's clear it up. We have to have a circuit in mind!

1. Series RLC circuit:



Kirchhoff says

$$V = V_R + V_C + V_L = iR + i/[j\omega C] + [j\omega L]\, i = i\,[\, R + 1/[j\omega C] + j\omega L\,]$$

$$= iZ \qquad \Rightarrow \qquad Z = R + 1/[j\omega C] + j\omega L = R - j/[\omega C] + j\omega L$$

If you write this as

$Z = R + jX$ R = resistance X = reactance

then

$X = \omega L - 1/(\omega C)$ // as on p 121 of Ref Dat EE book

// $X_C = 1/\omega C$ appears p 47 of my APL book
// $X_L = \omega L$ appears page 23 of same

So making these two definitions, we get

$X = X_L - X_C = \omega L - 1/(\omega C)$ $\qquad$ $Z = R + jX_L - jX_C$

This agrees with current wiki on reactance

Resonance occurs when X = 0 and your circuit then has Z = C.

$X_L = X_C \quad => \quad \omega L = 1/\omega C \quad => \quad \omega^2 = 1/(LC)$

You could define $Y = Z^{-1}$ for this circuit, but it won't be purdy:

$$Y = Z^{-1} = \frac{1}{R+jX} = \frac{1}{R+jX}\frac{R-jX}{R-jX} = \frac{R-jX}{R^2+X^2} = \frac{R}{R^2+X^2} - j\frac{X}{R^2+X^2} = G + jB$$

$$G = \text{conductance} = \frac{R}{R^2+X^2} \qquad B = \text{susceptance} = -\frac{X}{R^2+X^2}$$

which agrees with wiki on admittance

$$Y = Z^{-1} = \frac{1}{R+jX} = \left(\frac{1}{R^2+X^2}\right)(R-jX) \qquad G = \Re(Y) = \frac{R}{R^2+X^2} \qquad B = \Im(Y) = -\frac{X}{R^2+X^2}$$

Notice that $G \neq 1/R$ for example.

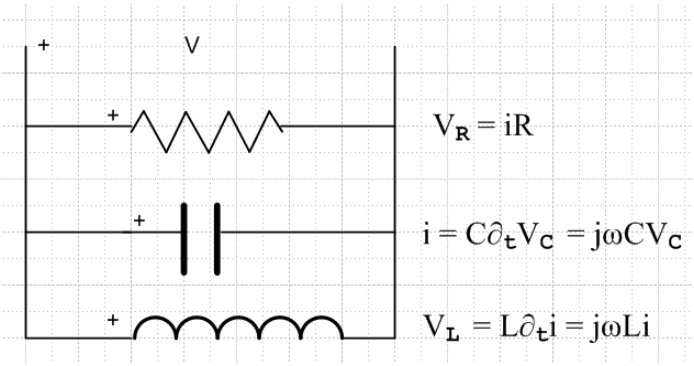
Rule:

$$V = i\,[\,R + 1/[j\omega C] + j\omega L\,] = i\,[Z_R + Z_C + Z_L] = i\,[R - jX_C + jX_L]$$

So you can just look at the circuit and write down the impedance by inspectino

$Z = Z_R + Z_C + Z_L = R - jX_C + jX_L$

2. Parallel RLC circuit:



Now $i = i_R + i_C + i_L = V_R/R + j\omega C V_C + V_L/[j\omega L] = V/R + j\omega C V + V/[j\omega L]$

$= V [1/R + j\omega C + 1/[j\omega L]]$

Then

$Y \equiv i/V = 1/R + j/X_C - j/X_L = 1/R + j (1/X_C - 1/X_L) = G + j B$

$B = (1/X_C - 1/X_L) = - 1/X_P$ // This is my Ref Data book p 120 definition of X_P

$X_P \equiv -X_C X_L/(X_L - X_C) = -(1/\omega C)(\omega L) / (- 1/\omega C + \omega L)$

$= \omega L / (1 - \omega^2 LC)$ // as on page 120 Ref Data

So the result here is fairly ugly

$Y = G + jB = 1/R + j X_C X_L/(X_L - X_C) = 1/R - j \omega L / (1 - \omega^2 LC)$

Then

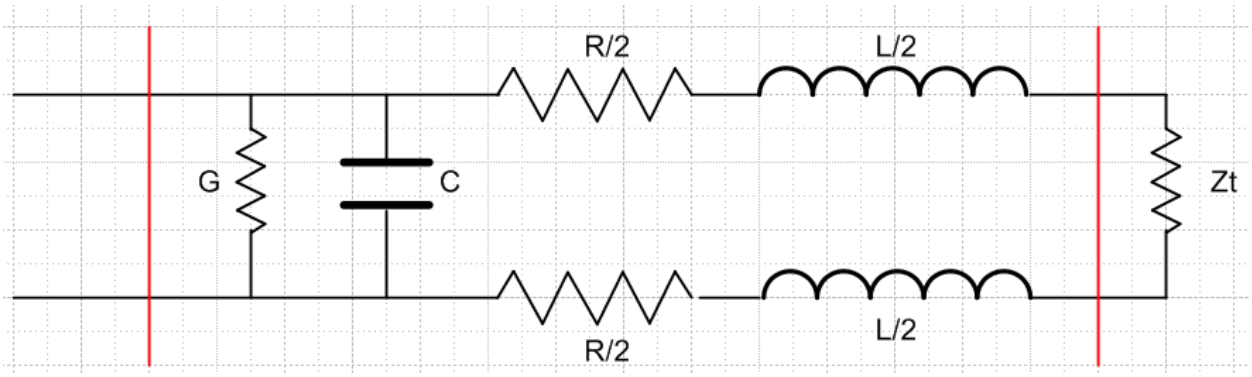
$$Z = Y^{-1} = \frac{1}{G+jB} = \frac{1}{G+jB}\frac{G-jB}{G-jB} = \frac{G-jB}{G^2+B^2} = \frac{G}{G^2+B^2} - j\frac{B}{G^2+B^2} = R + jX$$

Rule:

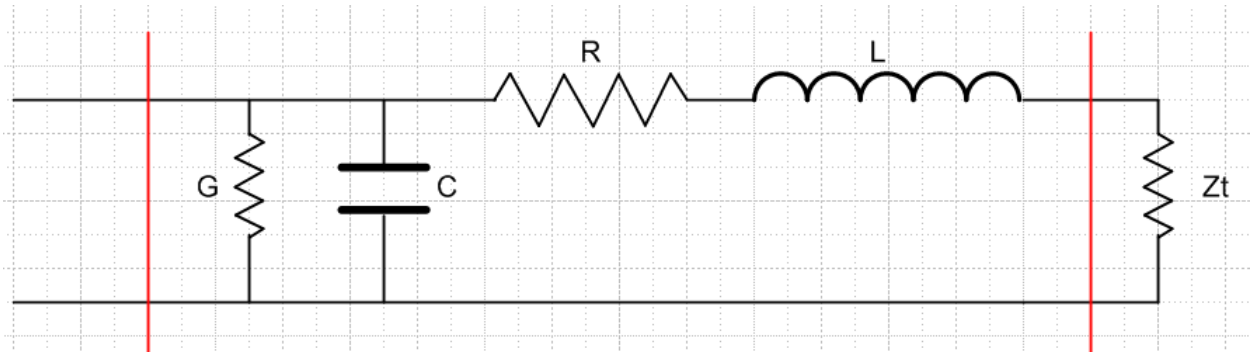
$Y = 1/R + j\omega C + 1/[j\omega L] = Y_R + Y_C + Y_L$

$= 1/R + j/X_C - j/X_L$

3. Transmission Line Circuit



We show a section of length dz between the red lines. R_t is some termination impedance. As you look in from the left, you want to see R_t. This picture is like that of King page 5. We can simplify to get



On the right we have a series RL circuit where we can neglect R compared to Z_t. Keeping R, we would say those three elements have

$$Z_1 = R + Z_t + jX_L$$

Now this is in parallel with G and C so overall looking in the left we have

$$Y = Y_G + Y_C + Y_1 = 1/R_d + j/X_C + [(R + Z_t) + jX_L]^{-1}$$

(a) Ignore R_d term and the R term

Now suppose we ignore$1/R_d$ and ignore R so we have

$$Y \approx j/X_C + [Z_t + jX_L]^{-1}$$

We want then to have

$$Z_{left} = Y^{-1} = Z_t$$

So then we have to solve

$$j/X_C + [Z_t + jX_L]^{-1} = 1/Z_t$$

or

$$(j/X_C)[Z_t + jX_L] + 1 = (1/Z_t)[Z_t + jX_L]$$

or

$$(j/X_C)[Z_t + jX_L] + 1 = 1 + (1/Z_t)[jX_L]$$

or

$$(j/X_C)[Z_t + jX_L] = (1/Z_t)[jX_L]$$

or

$$(1/X_C)[Z_t + jX_L] = (1/Z_t)[X_L]$$

We can then solve this for Z_t, but it is a quadratic in Z_t

$$(1/X_C)[Z_t^2 + jX_LZ_t] = [X_L]$$

or

$$Z_t^2 + jX_LZ_t = [X_CX_L]$$

or

$$Z_t^2 + (jX_L)Z_t + (-X_CX_L) = 0$$

So then

$$Z_t = [-jX_L \pm \sqrt{-X_L^2 + 4X_CX_L}]/2$$

Now throw in the values to get

$$Z_t = [-j\omega L \pm \sqrt{\omega^2L^2 + 4\omega L/\omega C}]/2$$

or

$$Z_t = [-j\omega L \pm \sqrt{\omega^2L^2 + 4L/C}]/2$$

Now the trick is to realize that L and C are each tiny since only for dz. In the limit

$$Z_t = \pm\sqrt{4L/C}/2 = +\sqrt{L/C}$$

This is the traditional result, as on Matick page 8.

(b) Keep R_d term and the R term

Now let's try again, not dropping anything. We have

$$Y = Y_G + Y_C + Y_1 = 1/R_d + j/X_C + [(R + Z_t) + jX_L]^{-1}$$

I suppose right off the bat you can replace the bracket with just Z_t since R and L are small. Then

$$Y = Y_G + Y_C + Y_1 = 1/R_d + j/X_C + 1/Z_t$$

Now for a dz section we can ignore R_d I suppose, and X_C as well, and then

$$Y = 1/Z_t.$$

But now we don't get any expression for Z_t, just an identity. You cannot throw things out too early. So OK, here we go

$$1/Z_t = 1/R_d + j/X_C + [(R + Z_t) + jX_L]^{-1}$$

or

$$(1/Z_t)[(R + Z_t) + jX_L] = (1/R_d + j/X_C)[(R + Z_t) + jX_L] + 1$$

or

$$1 + (1/Z_t)[(R + jX_L] = (1/R_d + j/X_C)[(R + Z_t) + jX_L] + 1$$

or

$$(1/Z_t)[(R + jX_L] = (1/R_d + j/X_C)[(R + Z_t) + jX_L]$$

or

$$(1/Z_t)[(R + jX_L] = (1/R_d + j/X_C)[R + jX_L + Z_t]$$

or

$$[R + jX_L] = (1/R_d + j/X_C)[(R + jX_L)Z_t + Z_t^2]$$

or

$$[R + jX_L]/(1/R_d + j/X_C) = [(R + jX_L)Z_t + Z_t^2]$$

or

$$Z_t^2 + b\,Z_t + c = 0$$

$$c = -[R + jX_L]/(1/R_d + j/X_C) \qquad b = (R + jX_L)$$

or

$$c = -[R + j\omega L]/(G + j\omega C) \qquad b = (R + j\omega L)$$

Then

$$b^2 - 4ac = (R + j\omega L)^2 + 4(R + j\omega L)/(G + j\omega C)$$

In the last term, everything is $\propto dz$, so the last term is order unity. The first term is $\propto dz^2$ so we drop it

$$b^2 - 4ac = 4(R + j\omega L)/(G + j\omega C)$$

Then

$$2Z_t = -(R + jX_L) + \sqrt{4(R + j\omega L)/(G + j\omega C)}$$

Again we drop the first terms since $\propto dz$ and then we get this famous result

$$Z_t = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

and this appears in my Radio Eng handbook on page 552. And of course it reduces to our previous result when $R = G = 0$.