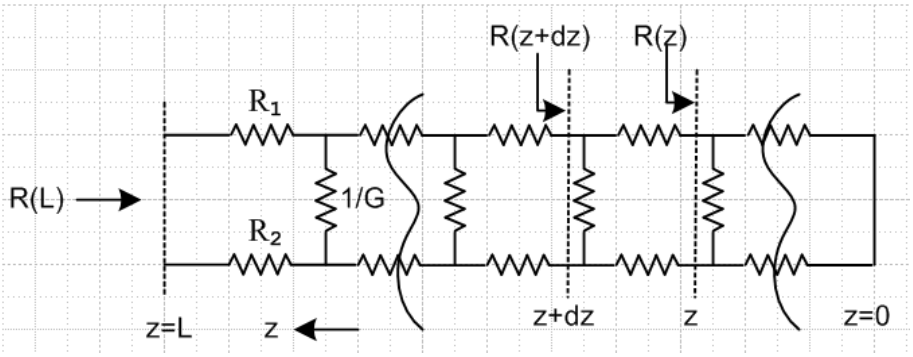
This is the Title PhL 3.26.05

This is now an exercise in Appendix K.

Reader Exercise: Consider the DC network transmission-line model shown below which is shorted at the right end, has length L, and z increases to the *left* as shown:



Since this is for ω = 0, the inductors and capacitors shown in Fig K.1 have been removed. Each section has length dz, the horizontal resistors have resistance $R_i dz$, and the vertical ones have conductance Gdz. The resistance looking in from the left at location z is R(z).

(a) Show that

$$R(z+dz) = \frac{1}{Gdz} \parallel (R(z) + R_3 dz) = \frac{[R(z) + R_3 dz]}{[R(z) + R_3 dz]Gdz + 1} \qquad \text{where } R_3 \equiv R_1 + R_2$$

Note that $\dim(R_i)$ = ohms/m, dim(G) = mhos/m, and dim[R(z)] = ohms.

(b) for very small dz show that

$$R(z+dz) \approx R(z) + (R_3 - G\,[R(z)]^2)dz$$

and therefore R(z) solves the following non-linear differential equation:

$$\frac{dR(z)}{dz} + G[R(z)]^2 = R_3 .$$

(c) Show that the general solution to this equation is

$$R(z) = \sqrt{R_3/G}\ \tanh(\sqrt{R_3 G}\ z + C) \qquad C = \text{constant}$$

and, since R(z) = 0, the solution is

$$R(z) = \sqrt{R_3/G}\ \tanh(\sqrt{R_3 G}\ z)$$

so

$$R(L) = \sqrt{R_3/G}\ \tanh\left(\sqrt{R_3 G}\ L\right)$$

(d) Show that

$$R(L) \approx \sqrt{R_3/G} \quad \text{if } L >> 1/\sqrt{R_3 G}$$

and that this agrees with (K.6) for an infinite line with $\omega = 0$. Thus, for large L the fact that the line is shorted at the right end makes no difference. Interpret the distance $1/\sqrt{R_3 G}$.

(e) Show that

$$R(L) \rightarrow R_3 L \qquad \text{as } G \rightarrow 0$$

which is the obviously correct limit for the figure shown above when G = 0.

Here is some Maple code related to item (c):

```
assume(G>0,R3>0);
eq := Diff(R(z),z) + G*R(z)^2 = R3;
```

$$eq := \left(\frac{\partial}{\partial z} R(z)\right) + G\,R(z)^2 = R3$$

```
R := z -> sqrt(R3/G)*tanh(sqrt(R3*G)*z+C);
```

$$R := z \rightarrow \sqrt{\frac{R3}{G}}\tanh\left(\sqrt{R3\,G}\,z + C\right)$$

```
eq;
```

$$\left(\frac{\partial}{\partial z}\sqrt{\frac{R3}{G}}\tanh\left(\sqrt{R3\,G}\,z + C\right)\right) + R3\tanh\left(\sqrt{R3\,G}\,z + C\right)^2 = R3$$

```
value(%);
```

$$\sqrt{\frac{R3}{G}}\left(1 - \tanh\left(\sqrt{R3\,G}\,z + C\right)^2\right)\sqrt{R3\,G} + R3\tanh\left(\sqrt{R3\,G}\,z + C\right)^2 = R3$$

```
simplify(%);
```

$$R3 = R3$$