Here I write the last two sections of Appendix L as analogs of the first two sections. All installed.

L.3 The potential of a line charge inside a thick dielectric cylindrical shell

In this section, we repeat everything done in Section L.1 in the 2D world instead of the 3D world. The 3D Poisson propagator $(1/4\pi\varepsilon_0)(1/r)$ becomes $(1/2\pi\varepsilon_0)\ln(1/r)$ as discussed in Appendix K. We reuse the same drawings, the first of which is

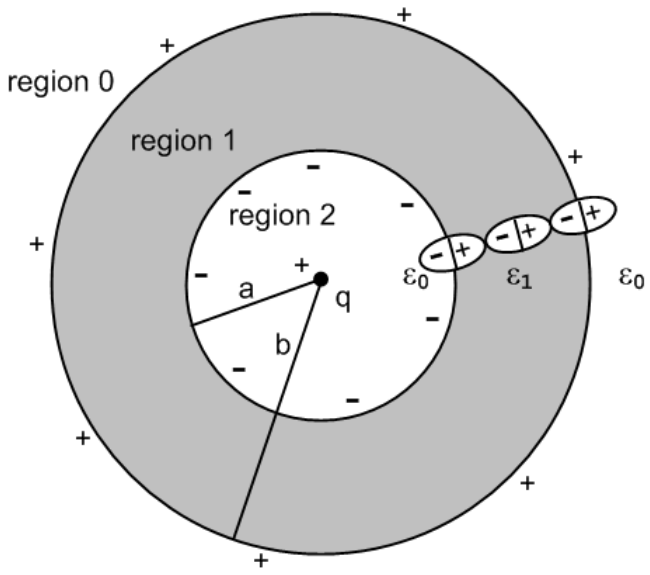


Fig L.1'

This is now a cross section of an infinite uniform cylindrical geometry. Quantity q is now a linear charge density with dimensions Coulombs/m. Rather that copy, paste and edit Section L.1, here we just show the altered equations and skip most of the words. The equation numbers are those of Section L.1 with a prime added. One difference encountered is that we must take $b\rightarrow R$ (a large value) rather than $b\rightarrow\infty$. As noted in Appendix K, a constant in a potential can be ignored even if it is infinite, and such constants do not appear in the field $\mathbf{E} = -\nabla\varphi$.

Ansatz potential forms: (to-be-determined constants are α, C, D)

$\varphi_0(r) = (1/2\pi\varepsilon_0)\, q \ln(1/r)$	$E_{r0}(r) = (1/2\pi\varepsilon_0)\, q/r$	region 0	
$\varphi_1(r) = (1/2\pi\alpha)\, q \ln(1/r) + C$	$E_{r1}(r) = (1/2\pi\alpha)\, q/r$	region 1	
$\varphi_2(r) = (1/2\pi\varepsilon_0)\, q \ln(1/r) + D$	$E_{r2}(r) = (1/2\pi\varepsilon_0)\, q/r$	region 2	(L.1.1)'

Continuity of φ at r=a and b:

$(1/2\pi\varepsilon_0)\, q \ln(1/b) = (1/2\pi\alpha)\, q \ln(1/b) + C$	region 0/1 boundary, r = b	
$(1/2\pi\varepsilon_0)\, q \ln(1/a) + D = (1/2\pi\alpha)\, q \ln(1/a) + C$	region 2/1 boundary, r = a	(L.1.3)'

Rule for E field normal components at a boundary:

$\varepsilon_0 E_{r0}(b) = \varepsilon_1 E_{r1}(b)$ region 0/1 boundary, r = b

$\varepsilon_0 E_{r2}(a) = \varepsilon_1 E_{r1}(a)$ region 2/1 boundary, r = a

or

$\varepsilon_0\, q/\,[2\pi\varepsilon_0 b] = \varepsilon_1\,(1/2\pi\alpha)\, q/b \Rightarrow 1 = \varepsilon_1/\alpha$

$\varepsilon_0\, q/\,[2\pi\varepsilon_0 a] = \varepsilon_1\,(1/2\pi\alpha)\, q/a \Rightarrow 1 = \varepsilon_1/\alpha$. (L.1.4)'

Restated continuity of φ with $\alpha = \varepsilon_1$:

$(1/2\pi\varepsilon_0)\, q \ln(1/b) = (1/2\pi\varepsilon_1)\, q \ln(1/b) + C$ region 0/1 boundary, r = b

$(1/2\pi\varepsilon_0)\, q \ln(1/a) + D = (1/2\pi\varepsilon_1)\, q \ln(1/a) + C$ region 2/1 boundary, r = a (L.1.5)'

Second equation minus first above:

$D + (1/2\pi\varepsilon_0)\, q(\ln(1/a) - \ln(1/b)) = (1/2\pi\varepsilon_1) q\, (\ln(1/a) - \ln(1/b))$

$\Rightarrow D + (1/2\pi\varepsilon_0)\, q \ln(b/a) = (1/2\pi\varepsilon_1)\, q \ln(b/a)$

Solution for the three constants:

$\alpha = \varepsilon$

$C = q \ln(1/b)(1/2\pi)\,(1/\varepsilon_0 - 1/\varepsilon_1)$

$D = q \ln(a/b)(1/2\pi)\,(1/\varepsilon_0 - 1/\varepsilon_1)$ (L.1.6)'

The potentials in the three regions are then

$\varphi_0(r) = (1/2\pi\varepsilon_0)\, q \ln(1/r)$

$\varphi_1(r) = (1/2\pi\varepsilon_1)\, q \ln(1/r) + (q/2\pi) \ln(1/b)\,(1/\varepsilon_0 - 1/\varepsilon_1)$

$\varphi_2(r) = (1/2\pi\varepsilon_0)\, q \ln(1/r) + (q/2\pi) \ln(a/b)\,(1/\varepsilon_0 - 1/\varepsilon_1)$ (L.1.7)'

while the fields are

$E_{r0} = (1/2\pi\varepsilon_0)\, q/r$

$E_{r1} = (1/2\pi\varepsilon_1)\, q/r$

$E_{r2} = (1/2\pi\varepsilon_0)\, q/r$. (L.1.8)'

The following Maple plots show the continuity of φ and the jumps in E_r at the boundaries

```
phi := piecewise(r>=0 and r<a,phi2,r>=a and r<b,phi1,r>=b,phi0):
Er := piecewise(r>=0 and r<a,Er2,r>=a and r<b,Er1,r>=b,Er0):
phi0 := q/(2*Pi*e0)*ln(1/r):
phi1 := q/(2*Pi*e1)*ln(1/r) + q/(2*Pi)*ln(1/b)*(1/e0-1/e1):
phi2 := q/(2*Pi*e0)*ln(1/r) + q/(2*Pi)*ln(a/b)*(1/e0-1/e1):
Er0 :=  q/(2*Pi*e0*r):
Er1 :=  q/(2*Pi*e1*r):
Er2 :=  q/(2*Pi*e0*r):
q := 1: a := 1: b := 2: e0 := 1: e1 := 2:
plot([phi,Er], r=0.5..5, color = [red,blue]);
```

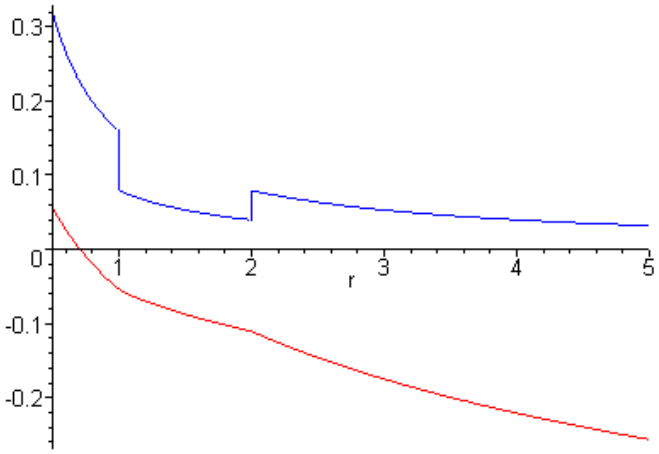


Fig L.2'

What about the (now linear) bound charge densities at r = a and r = b?

$\mathbf{P} = (\varepsilon-\varepsilon_0)E_r\hat{\mathbf{r}}$

$P_{r1}(r) = (\varepsilon_1-\varepsilon_0)E_{r1}(r)\ \theta(r>a)\theta(r<b)$ // points radially outward since $\varepsilon_1 > \varepsilon_0$ (L.1.9)'

From (1.1.11) the polarization charge density is then

$\rho_{pol} = -\ \mathrm{div}\ \mathbf{P}$ *(1.1.11)*

so in cylindrical coordinates,

$\rho_{pol}(r) = -\ [r^{-1}\partial_r(rP_r) + r^{-1}\partial_\theta P_\theta + \partial_z P_z]$

$= -\ r^{-1}\partial_r(rP_r)$

$= -\ r^{-1}\partial_r(r[(\varepsilon_1-\varepsilon_0)E_{r1}(r)\ \theta(r>a)\theta(r<b)])$

$= -\ r^{-1}\partial_r(r[(\varepsilon_1-\varepsilon_0)\ (1/2\pi\varepsilon_1)\ q/r\ \theta(r>a)\theta(r<b)])$

$= -\ q(\varepsilon_1-\varepsilon_0)\ (1/2\pi\varepsilon_1)r^{-1}\partial_r[\theta(r-a)\theta(b-r)]$

$$= - q(\varepsilon_1-\varepsilon_0)\,(1/2\pi\varepsilon_1) r^{-1}\,[\,\delta(r-a)\,\theta(b-r) - \theta(r-a)\delta(r-b)\,]$$

$$= - q(\varepsilon_1-\varepsilon_0)\,(1/2\pi\varepsilon_1)\,[\,\delta(r-a)/a - \delta(r-b)/b\,]\ . \qquad \text{(L.1.10)'}$$

We may then read off the bound *linear* charge densities at r = a and b

$$\sigma_{inner} = - (\varepsilon_1-\varepsilon_0)\,(1/2\pi\varepsilon_1)(q/a) \qquad \text{// Coulombs/m}$$
$$\sigma_{outer} = \ (\varepsilon_1-\varepsilon_0)\,(1/2\pi\varepsilon_1)(q/b) \qquad \text{(L.1.11)'}$$

$$Q_{inner} = \oint ds\ \sigma_{inner} = - (\varepsilon_1-\varepsilon_0)\,(1/2\pi\varepsilon_1)(q/a) * 2\pi a = \ - (1-\varepsilon_0/\varepsilon_1)\ q$$

$$Q_{outer} = \oint ds\ \sigma_{outer} = (\varepsilon_1-\varepsilon_0)\,(1/2\pi\varepsilon_1)(q/b^{b}) * 2\pi b = \ (1-\varepsilon_0/\varepsilon_1)\ q \qquad \text{(L.1.12)'}$$

$$Q = (1-\varepsilon_0/\varepsilon_1)\ q \qquad \text{// exactly the same equation as in the 3D case} \qquad \text{(L.1.13)'}$$

Here Q is the total charge/m on the outer surface of the cylindrical shell at r = b, and -Q is the same thing at r = a. Recall that q is the charge/m of the central linear line charge.

L.4 Limits of the Previous Problem

(a) Line charge in an infinite cylindrical hole in a dielectric

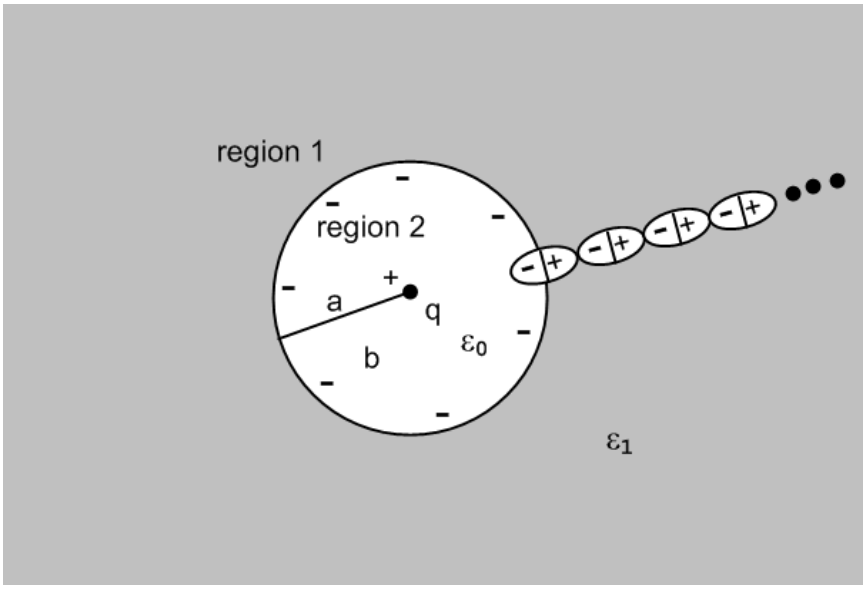


Fig L.3'

The potentials and fields shown in (L.1.7)' and (L.1.8)' are then, taking b→R (some large value)

$$\varphi_1(r) = (1/2\pi\varepsilon_1)\ q\ \ln(1/r) + (q/2\pi)\ \ln(1/R)\ (1/\varepsilon_0-1/\varepsilon_1)$$
$$\varphi_2(r) = (1/4\pi\varepsilon_0)\ q/r\ - (q/4\pi a)(1/\varepsilon_0-1/\varepsilon_1) \qquad \text{(L.2.1)'}$$

while the fields are

$$E_{r1} = (1/2\pi\varepsilon_1)\, q/r$$
$$E_{r2} = (1/2\pi\varepsilon_0)\, q/r \quad . \qquad \text{(L.2.2)'}$$

The induced bound charge density σ at r = a is still given by

$$\sigma = -(\varepsilon_1-\varepsilon_0)\,(1/2\pi\varepsilon_1)(q/a)$$
$$-Q = (1-\varepsilon_0/\varepsilon_1)\, q \qquad \text{(L.2.3)'}$$

where -Q is the total bound charge on the r=a surface.

(b) Line charge embedded in an infinite dielectric cylinder

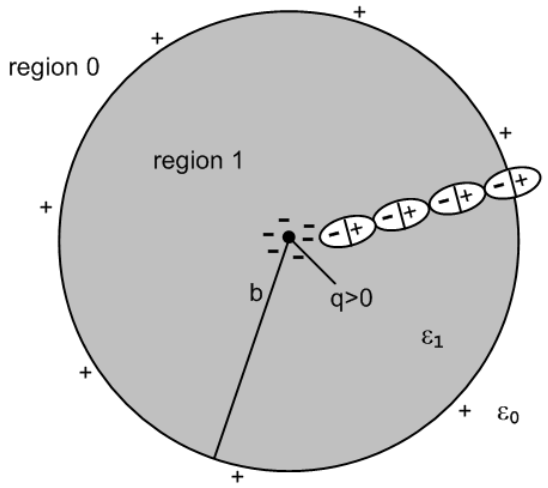


Fig L.5'

The potentials and fields shown in (L.1.7)' and (L.1.8)' are then, taking a→0

$$\varphi_0(r) = (1/2\pi\varepsilon_0)\, q \ln(1/r)$$
$$\varphi_1(r) = (1/2\pi\varepsilon_1)\, q \ln(1/r) + (q/2\pi) \ln(1/b)\,(1/\varepsilon_0 - 1/\varepsilon_1) \qquad \text{(L.2.4)'}$$

while the fields are

$$E_{r0} = (1/2\pi\varepsilon_0)\, q/r$$
$$E_{r1} = (1/2\pi\varepsilon_1)\, q/r \quad = \quad (1/2\pi\varepsilon_0)\,[q(\varepsilon_0/\varepsilon_1)]\,/\,r \qquad \text{(L.2.5)'}$$

where the last expression shows the "shielded charge interpretation".

(c) Line charge embedded in an infinite dielectric

We now take b→R (a large value) in the previous limit to get:

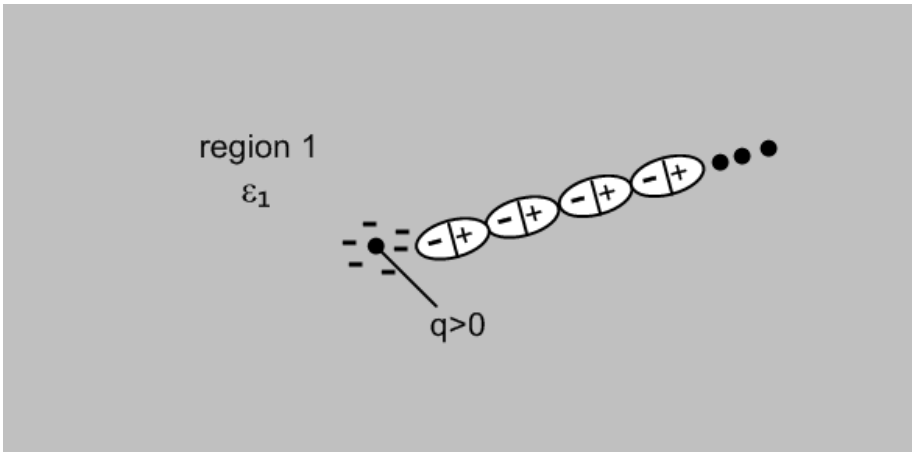


Fig L.6'

There is only one region left and from (L.2.4)' and (L.2.5') we get

$$\varphi_1(r) = (1/2\pi\varepsilon_1)\, q \ln(1/r) + (q/2\pi) \ln(1/R)\, (1/\varepsilon_0 - 1/\varepsilon_1) \qquad \text{(L.2.6)'}$$

while the field is

$$E_{r1} = (1/2\pi\varepsilon_1)\, q/r \quad = \quad (1/2\pi\varepsilon_0)\, [q(\varepsilon_0/\varepsilon_1)] / r \qquad \text{(L.2.7)'}$$

where the last expression shows the "shielded charge interpretation".