Point charge in two dielectrics, Part 2 PhL 10.24.13

See notes in the Part 3 version.

There is more stuff I need to know about this problem before I can attack my two King logjam question.

A Wrong Assumption: First of all, since div **D** = ρ = the free charge only, you would think that the field **D** would be

$$\mathbf{D}_1 = \mathbf{D}_2 = q\,(1/4\pi)\,(1/r)\,\hat{\mathbf{r}}$$

since **D** does not "see" the polarization charge at the boundary. Here we place the Jackson region 1 point charge at a spherical system origin.

We then know that

$\mathbf{P}_1 = \varepsilon_0\chi_1\,\mathbf{E}_1$ in region 1
$\mathbf{P}_2 = \varepsilon_0\chi_2\,\mathbf{E}_2$ in region 2

$\mathbf{D}_1 = \varepsilon_1\mathbf{E}_1$ in region 1
$\mathbf{D}_2 = \varepsilon_2\mathbf{E}_2$ in region 2

But since $\mathbf{D}_1 = \mathbf{D}_2 = \mathbf{D}$ = the same in both regions, we seem to get

$q\,(1/4\pi)\,(1/r)\,\hat{\mathbf{r}} = \varepsilon_1\mathbf{E}_1$ in region 1
$q\,(1/4\pi)\,(1/r)\,\hat{\mathbf{r}} = \varepsilon_2\mathbf{E}_2$ in region 2

But this is NOT what we get. The reason is that in region 1 we don't get $\mathbf{E}_1 = [...]\hat{\mathbf{r}}$, but instead we know the E field lines are curved. So what have I done wrong here?

What is wrong is this. Knowing that div **D** = ρ does not determine **D**. This is just a condition on **D**. The other major condition is curl **E** = **0** for a static situation. And we have to make sure things are OK at boundaries, and THEN we have a solution.

I said that **D** does not see polarization charge at a boundary. That is true, but we have to incorporate boundary conditions. Recall the integral form of our two rules,

$$\text{div}\,\mathbf{D} = \rho \quad \Leftrightarrow \quad \int_V \rho\,dV = \int_S \mathbf{D}\bullet d\mathbf{A} \qquad (1.1.13)$$

$$\text{curl}\,\mathbf{E} = 0 \quad \Leftrightarrow \quad \oint \mathbf{E}\bullet d\mathbf{s} = 0 \qquad (1.1.17)$$

This means that $E_{||}$ is the same on both sides of a boundary, while $D_\perp$ is the same on both sides. For our little Jackson problem that means

$$E_{||}^{(1)} = E_{||}^{(2)}$$

$$\varepsilon_1 E_\perp^{(1)} = \varepsilon_2 E_\perp^{(2)}$$

or

$$(1/\varepsilon_1) D_{||}^{(1)} = (1/\varepsilon_2) D_{||}^{(2)}$$
$$D_\perp^{(1)} = D_\perp^{(2)}$$

Thus, even thought D does not see polarization charge at a boundary, it does not simply "pass through" a boundary as I suggested above. There is refraction of D at the boundary just as with E !

Now how would you approach solving the D problem? I don't really know! I only know how to approach the φ problem and $\mathbf{E} = -\nabla\varphi$ so I am better at doing the E problem.

My "cylindrical atoms" solution to the problem was this:

$$\varphi_1(\rho,z) = (1/4\pi\varepsilon_1) \int_0^\infty dk\, e^{-k|z|} J_0(k\rho) + \int_0^\infty dk\, A_k\, e^{kz} J_0(k\rho)$$

$$\varphi_2(\rho,z) = \int_0^\infty dk\, B_k\, e^{-kz}\, J_0(k\rho) \qquad R = \sqrt{\rho^2 + s^2}$$

$$E_{z1} = \partial_z \varphi_1 = (1/4\pi\varepsilon_1) \int_0^\infty dk\, (\partial_z e^{-k|z|}) J_0(k\rho) + \int_0^\infty dk\, A_k\, k\, e^{kz} J_0(k\rho)$$

$$E_{z2} = \partial_z \varphi_2 = \int_0^\infty dk\, B_k\, (-k)\, e^{-kz}\, J_0(k\rho)$$

$$E_{\rho 1} = \partial_\rho \varphi_1 = (1/4\pi\varepsilon_1) \int_0^\infty dk\, k\, e^{-k|z|} J_0'(k\rho) + \int_0^\infty dk\, kA_k\, e^{kz} J_0'(k\rho)$$

$$E_{\rho 2} = \partial_\rho \varphi_2 = \int_0^\infty dk\, kB_k\, e^{-kz} J_0'(k\rho)$$

$$A_k = (1/4\pi\varepsilon_1) e^{-2ks} [(\varepsilon_1 - \varepsilon_2)/(\varepsilon_1 + \varepsilon_2)]$$
$$B_k = (1/4\pi\varepsilon_1)\, [2\varepsilon_1 / (\varepsilon_1 + \varepsilon_2)] \equiv B$$

<u>Ratio.</u> For fun, let's look at the ratio of $E_{z2}/E_{\rho 2}$

$$E_{z2}/E_{\rho 2} = \int_0^\infty dk\, B_k\, (-k)\, e^{-kz}\, J_0(k\rho) \;/\; \int_0^\infty dk\, k\, B_k\, e^{-kz} J_0'(k\rho)$$

$$= - \int_0^\infty dk\, k\, e^{-kz}\, J_0(k\rho) \;/\; \int_0^\infty dk\, k\, e^{-kz} J_0'(k\rho)$$

where the B_k just cancel, being constants as shown above. If we use $\rho^2 + z^2 = r^2$ to eliminate either ρ or z, the solution claims that this ratio is a constant independent of r ! That result seems completely non-obvious looking at the integrals. Maybe write this ratio this way

$$E_{z2}/E_{\rho 2} = \partial_z \varphi_2 \;/\; \partial_\rho \varphi_2 \qquad \text{where } \varphi_2 = B \int_0^\infty dk\, e^{-kz}\, J_0(k\rho)$$

For $z > 0$, we can see that $\int_0^\infty dk\, e^{-kz}\, J_0(k\rho) = 1/R$ for a unit point charge at the origin, but this is true only for $z > 0$!! That is why we get the lines *there* all zeroing in on the location of q.

So it is quite easy to obtain the image charge solution from my results above.

Free charge on the boundary. What about the surface charge distribution? We get that from

$$\text{div } \mathbf{D} = \rho \qquad \Leftrightarrow \qquad \int_V \rho\, dV = \int_S \mathbf{D} \bullet d\mathbf{A} \tag{1.1.13}$$

If we apply this to a tiny pillbox straddling the boundary, we find that

$$n\, dA = D_{z2} dA - D_{z1} dA \quad \Rightarrow \quad n = D_{z2} - D_{z1}$$

$$\Rightarrow \; n = \varepsilon_2 E_{z2} - \varepsilon_1 E_{z1}$$

where from above, at $z = s$,

$$E_{z1} = \partial_z \varphi_1 = (1/4\pi\varepsilon_1) \int_0^\infty dk\, (-ke^{-ks}) J_0(k\rho) + \int_0^\infty dk\, A_k\, k\, e^{ks}\, J_0(k\rho)$$

$$E_{z2} = \partial_z \varphi_2 = B \int_0^\infty dk\, (-k)\, e^{-ks}\, J_0(k\rho)$$

$$A_k = (1/4\pi\varepsilon_1) e^{-2ks} [(\varepsilon_1-\varepsilon_2)/(\varepsilon_1+\varepsilon_2)] = (1/4\pi\varepsilon_1) e^{-2ks}\, f \qquad f = [(\varepsilon_1-\varepsilon_2)/(\varepsilon_1+\varepsilon_2)]$$

$$B = (1/4\pi\varepsilon_1)\, [2\varepsilon_1 / (\varepsilon_1+\varepsilon_2)] = (1/4\pi\varepsilon_1)\, g \qquad g = [2\varepsilon_1 / (\varepsilon_1+\varepsilon_2)]$$

Therefore

$$E_{z1} = \partial_z \varphi_1 = -(1/4\pi\varepsilon_1) \int_0^\infty dk\, k\, e^{-ks} J_0(k\rho) + (1/4\pi\varepsilon_1)\, f \int_0^\infty dk\, k\, e^{-ks}\, J_0(k\rho)$$

$$= (1/4\pi\varepsilon_1)\, [f - 1] \int_0^\infty dk\, k\, e^{-ks}\, J_0(k\rho)$$

$$E_{z2} = (1/4\pi\varepsilon_1)\, [-g] \int_0^\infty dk\; k\, e^{-ks}\, J_0(k\rho)$$

Then the surface charge density seems to be this

$$n = \varepsilon_2 E_{z2} - \varepsilon_1 E_{z1}$$

$$= (1/4\pi\varepsilon_1)\, [-\varepsilon_2 g - \varepsilon_1(f-1)] \int_0^\infty dk\, k\; e^{-ks}\, J_0(k\rho)$$

$$= -(1/4\pi\varepsilon_1)\, [\varepsilon_2 g + \varepsilon_1(f-1)] \int_0^\infty dk\, k\; e^{-ks}\, J_0(k\rho)$$

Meanwhile, Maple tells us that $[\varepsilon_2 g + \varepsilon_1(f-1)] = 0$ so our result is that n = 0. This is the correct result for the free charge at the boundary!

```
f := (e1-e2)/(e1+e2);
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$$f := \frac{e1 - e2}{e1 + e2}$$

```
g := 2*e1/(e1+e2);
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$$g := 2\,\frac{e1}{e1 + e2}$$

```
t := e2*g + e1*(f-1):simplify(%);
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$$0$$

I think the integral shown is a simple elementary function but hold on that for a moment.

Polarization charge on the boundary. Now let's find the charge that E sees!

$$\text{div } \mathbf{E} = \rho/\varepsilon \quad \Leftrightarrow \quad \int_V (\rho/\varepsilon)\, dV = \int_S \mathbf{E} \bullet d\mathbf{A} \quad [****] \qquad (1.1.14)$$

If we apply this to a tiny pillbox straddling the boundary, what do we do about ε? Suppose we put half the charge into each half space, just as something to try. Then let ε_{av} be the average of the two, so

$$(n/\varepsilon_1 + n/\varepsilon_2)\,dA = (E_{z2} - E_{z1})\,dA$$

but this is baloney, you could use any fraction you wanted. So I guess the problem is that *you cannot find* the total charge ρ from the above conditions! You have to involve the P vector somehow.

So let's accept Jackson's claim that

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}[\rho - \nabla \cdot \mathbf{P}] \qquad (4.33)$$

where ρ is the free charge, and therefore

$$\rho_{pol} = -\,\text{div } \mathbf{P}$$

when then leads to

$$\text{div } \mathbf{P} = -\rho_{pol} \quad \Leftrightarrow \quad -\int_V \rho_{pol}\, dV = \int_S \mathbf{P} \bullet d\mathbf{A}$$

Now apply this to our pillbox to get (see Attempt #2 picture)

$$-\,n_{pol}\,dA = [P_{2z} - P_{1z}]\,dA$$

or

$$-\,n_{pol} = P_{2z} - P_{1z}$$

Now we know that

$$\mathbf{P}_1 = \varepsilon_0\chi_1\mathbf{E}_1 \quad = \varepsilon_0(\varepsilon_1/\varepsilon_0 - 1)\,\mathbf{E}_1 \;= (\varepsilon_1-\varepsilon_0)\,\mathbf{E}_1$$
$$\mathbf{P}_2 = \varepsilon_0\chi_2\mathbf{E}_2$$

and therefore

$$-n_{pol} = \varepsilon_0\chi_2 E_{2z} - \varepsilon_0\chi_1 E_{1z}$$

or

$$-n_{pol}/\varepsilon_0 = \chi_2 E_{2z} - \chi_1 E_{1z}$$

Meanwhile, our boundary condition on E is this:

$$\varepsilon_1 E_{\perp}^{(1)} = \varepsilon_2 E_{\perp}^{(2)} \qquad \Rightarrow \varepsilon_1 E_{1z} = \varepsilon_2 E_{2z}$$

Since E_{z2} is my simpler known form, I will keep it so that then

$$-n_{pol}/\varepsilon_0 = \chi_2 E_{2z} - \chi_1(\varepsilon_2/\varepsilon_1)E_{2z} = [\,\chi_2 - \chi_1(\varepsilon_2/\varepsilon_1)]\,E_{2z}$$

But we know that $\epsilon/\epsilon_0 = 1 + \chi_e$ so

$$\chi_1 = \varepsilon_1/\varepsilon_0 - 1$$
$$\chi_2 = \varepsilon_2/\varepsilon_0 - 1$$

Then

$$-n_{pol}/\varepsilon_0 = [\,\chi_2 - \chi_1(\varepsilon_2/\varepsilon_1)]\,E_{2z} = [-(\varepsilon_2-\varepsilon_1)/\varepsilon_1]\,E_{2z}$$

where Maple did this little piece of algebra

```
restart;
x1 := e1/e0-1;
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$$x1 := \frac{e1}{e0} - 1$$

```
x2 := e2/e0-1;
```

$$x2 := \frac{e2}{e0} - 1$$

```
w := x2-x1*(e2/e1);
```

$$w := \frac{e2}{e0} - 1 - \frac{\left(\frac{e1}{e0} - 1\right) e2}{e1}$$

```
simplify(%);
```

$$-\frac{e1 - e2}{e1}$$

My conclusion then is that

$$n_{pol}/\varepsilon_0 = [(\varepsilon_2 - \varepsilon_1)/\varepsilon_1]\, E_{2z}$$

$$= [(\varepsilon_2 - \varepsilon_1)/\varepsilon_1]\, B \int_0^\infty dk\,(-k)\, e^{-ks}\, J_0(k\rho)$$

$$= [(\varepsilon_2 - \varepsilon_1)/\varepsilon_1]\, (1/4\pi\varepsilon_1)\, [2\varepsilon_1 / (\varepsilon_1+\varepsilon_2)] \int_0^\infty dk\,(-k)\, e^{-ks}\, J_0(k\rho)$$

$$= -(\varepsilon_2-\varepsilon_1)]\, (1/2\pi\varepsilon_1)\, [1 / (\varepsilon_1+\varepsilon_2)] \int_0^\infty dk\, k\, e^{-ks}\, J_0(k\rho)$$

$$= (1/2\pi\varepsilon_1)\, (\varepsilon_1-\varepsilon_2)/(\varepsilon_1+\varepsilon_2) \int_0^\infty dk\, k\, e^{-ks}\, J_0(k\rho)$$

Now let's go do this integral. Maple cannot do it, so I will go look it up in GR7.

2. $$\int_0^\infty e^{-\alpha x} J_\nu(\beta x) x^{\nu+1}\, dx = \frac{2\alpha(2\beta)^\nu\, \Gamma\left(\nu+\frac{3}{2}\right)}{\sqrt{\pi}\,(\alpha^2+\beta^2)^{\nu+\frac{3}{2}}} \qquad [\mathrm{Re}\,\nu > -1, \quad \mathrm{Re}\,\alpha > |\mathrm{Im}\,\beta|]$$

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So set $\alpha = s$, $\nu=0$, $\beta = \rho$ and we meet the conditions and the result is then

$$2s(2\rho)^0\, \Gamma(3/2)/[\sqrt{\pi}\,(s^2+\rho^2)^{3/2}] = 2s\{\sqrt{\pi}\,/2)/[\sqrt{\pi}\,(s^2+\rho^2)^{3/2}] = s / (s^2+\rho^2)^{3/2}$$

and therefore I conclude that

$$n_{pol}/\varepsilon_0 = (q/2\pi\varepsilon_1)\, (\varepsilon_1-\varepsilon_2)/(\varepsilon_1+\varepsilon_2)\, s / (s^2+\rho^2)^{3/2}$$

and finally

$$n_{pol} = -(q\varepsilon_0/2\pi\varepsilon_1)\, (\varepsilon_2-\varepsilon_1)/(\varepsilon_2+\varepsilon_1)\, s/ (s^2+\rho^2)^{3/2}$$

Jackson's result was this

$$\sigma_{\text{pol}} = -\frac{q}{2\pi}\,\frac{\epsilon_0(\epsilon_2 - \epsilon_1)}{\epsilon_1(\epsilon_2 + \epsilon_1)}\,\frac{d}{(\rho^2 + d^2)^{3/2}} \qquad (4.47)$$

so we are in perfect agreement. Hurray!

Question: Given some polarization charge ρ_{pol} , how do you compute its electric field?

Answer: Look back at Jackson's claim

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}[\rho - \nabla \cdot \mathbf{P}] \qquad (4.33)$$

If we are in a medium of $\varepsilon \neq \varepsilon_0$, the claim is that we have

div **E** = $\rho/\varepsilon_0 + \rho_{pol}/\varepsilon_0$ where ρ_{pol} = - div **P**.

Thus, we must propagate both the charges with $1/4\pi\varepsilon_0$, not with $1/4\pi\varepsilon$!! It's just that when we add up these two sources, we find

div **D** = ρ and div **E** = ρ/ε

where ρ is the total charge. Let's prove this claim. Start with the above

div **E** = $\rho/\varepsilon_0 + \rho_{pol}/\varepsilon_0 = \rho/\varepsilon_0 - 1/\varepsilon_0$ div **P**

or

$\varepsilon_0$div **E** = ρ – div **P**

or

div(ε_0**E** + **P**) = ρ

Then define as Jackson does, $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$, so that

div(**D**) = ρ

Then since **D** = ε **E** we get

div(**E**) = ρ/ε

where ρ is only the free charge. So if you ignore the polarization charge, then you compute a potential by using $1/4\pi\varepsilon R$ as the propagator. But if don't ignore it, they you propagate each piece as $1/4\pi\varepsilon_0 R$. This is an important point that I missed earlier.

So, when we talk about the charge on our boundary as being just polarization charge, it has to propagate with $1/4\pi\varepsilon_0 R$ in either region 1 or region 2!

Compute E field due to polarization charge.

Here is my claim, words follow:

$\mathbf{E} = E_z \hat{\mathbf{z}}$ $E_z = \int [2\pi\rho d\rho] n(\rho) \{ 1/ (z^2 +$