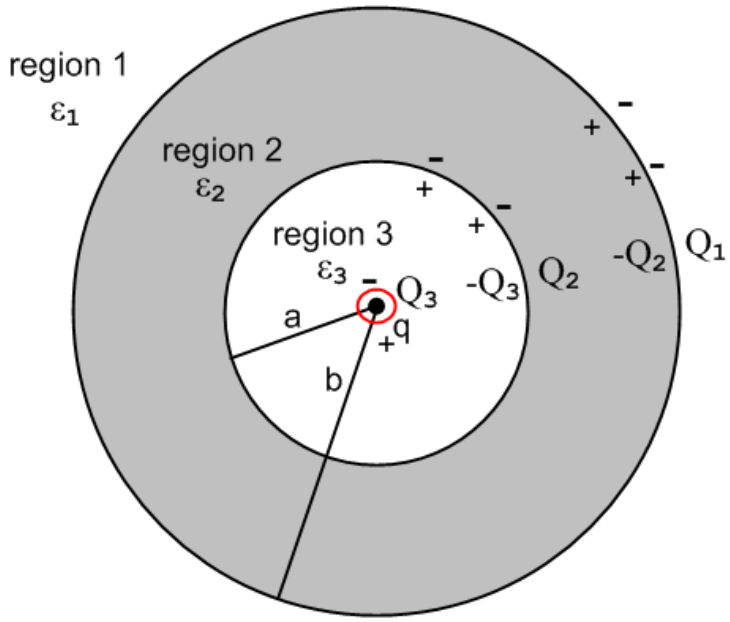
Charge in Sphere Problem part 2 PhL 11.6.13

This stuff all ended up in Appendix L!

I want to do a more general problem than I did in part 1.

Problem: For this spherical picture with positive q at the center, compute everything. None of the media is conducting.



We can add the layers at the two boundaries to say

$Q_{23} = Q_2 - Q_3$
$Q_{12} = Q_1 - Q_2$

The charge on the great sphere will be $-Q_1$. We will compute all these charges below as part of the solution to this problem.

Solution: We know that $\varphi = \varphi(r)$ and $\mathbf{E} = E_r\hat{\mathbf{r}}$ in all three regions. The first issue is what are the general forms of the solution in each region? In my first go at this, I allowed linear terms Ar in φ_3 and Dr in φ_2, but I now realize they cannot be present because the field must always have the form K/r^2 due to spherical symmetry. But I leave them in since that is how I started, and then A = D = 0 pretty soon.

One BC I got wrong the first time was the first term in φ_3. If you are close to q in region 3, then you can use the result of the LAST document to conclude that the first term must be $(1/4\pi\varepsilon_3)q/r$, and when I did this problem the first time I wrongly used ε_0 there. You HAVE to do that other problem first in order to know about this BC.

Using obvious atomic forms we can write in each region

$\varphi_3 = (1/4\pi\varepsilon_3)\, q/r + Ar + B$ // has right small r limit, I think A = 0 but lets see

$\varphi_2 = C\, q/r + Dr + E$ // just trying out a few known atomic forms.

$\varphi_1 = F\, q/r$ // since very far away we expect to have 1/r drop-off

The continuity conditions on φ are

at r = b: $F\, q/b = C\, q/b + Db + E$ 1 meets 2

at r = a: $C\, q/a + Da + E = (1/4\pi\varepsilon_3)\, q/a + Aa + B$ 2 meets 3

This is only 2 conditions on 6 parameters. But we now have to do field continuity.
Compute the three fields $-\mathbf{E} = \nabla\varphi$:

$-E_3 = -(1/4\pi\varepsilon_3)q/r^2 + A$

$-E_2 = -C\, q/r^2 + D$

$-E_1 = -Fq/r^2$

The field conditions are

at r = b: 1:2 $\varepsilon_1 E_{n1} = \varepsilon_2 E_{n2}$ $-\varepsilon_1\, Fq/b^2 = \varepsilon_2[\, -C\, q/b^2 + D]$

at r = a: 2:3 $\varepsilon_3 E_{n3} = \varepsilon_2 E_{n2}$ $\varepsilon_3[-(1/4\pi\varepsilon_3)\, q/a^2 + A] = \varepsilon_2[\, -C\, q/a^2 + D]$

We now have 4 conditions on 6 constants. There are no other conditions I can think of. I suspect that A and D both vanish, so lets assume A = 0 and D = 0 and solve the four resulting equations which are then

$F\, q/b = C\, q/b + E$ (1)

$C\, q/a + E = (1/4\pi\varepsilon_3)\, q/a + B$ (2)

$-\varepsilon_1\, Fq/b^2 = \varepsilon_2[\, -C\, q/b^2]$ => $-\varepsilon_1 F = -\varepsilon_2 C$ (3)

$\varepsilon_3[-(1/4\pi\varepsilon_3)\, q/a^2] = \varepsilon_2[\, -C\, q/a^2]$ => $\varepsilon_3\, [(1/4\pi\varepsilon_3)] = \varepsilon_2 C$ (4)

Variables are C, F, E, B. The last equation tells us C, and then (3) tells us F

$C = [(1/4\pi\varepsilon_3)]\, (\varepsilon_3/\varepsilon_2) = (1/4\pi\varepsilon_2)$

$-F = -\,(\varepsilon_2/\varepsilon_1)C = -\,[(1/4\pi\varepsilon_3)]\, (\varepsilon_3/\varepsilon_2)\, (\varepsilon_2/\varepsilon_1) = -\,(1/4\pi\varepsilon_3)\, (\varepsilon_3/\varepsilon_1) = -\,(1/4\pi\varepsilon_1)$

which I summarize as

$$C = +(1/4\pi\varepsilon_3)(\varepsilon_3/\varepsilon_2) = (1/4\pi\varepsilon_2)$$
$$F = +(1/4\pi\varepsilon_3)(\varepsilon_3/\varepsilon_1) = (1/4\pi\varepsilon_1)$$

Then equation (1) says

$$E = (q/b)(F-C) = -(q/b)(C-F) = -(q/b)(1/4\pi\varepsilon_3)[(\varepsilon_3/\varepsilon_2) - (\varepsilon_3/\varepsilon_1)] = -(q/b)(1/4\pi)(1/\varepsilon_2 - 1/\varepsilon_1)$$

Finally we can solve equation (3) for B

$$B = -(1/4\pi\varepsilon_3)\, q/a + C\, q/a + E$$

$$= -(1/4\pi\varepsilon_3)\, q/a + (1/4\pi\varepsilon_3)(\varepsilon_3/\varepsilon_2)\, q/a - (q/b)(1/4\pi\varepsilon_3)[(\varepsilon_3/\varepsilon_2) - (\varepsilon_3/\varepsilon_1)]$$

$$= (1/4\pi\varepsilon_3)\{ -q/a + (\varepsilon_3/\varepsilon_2)\, q/a - (q/b)[(\varepsilon_3/\varepsilon_2) - (\varepsilon_3/\varepsilon_1)] \}$$

$$= (1/4\pi\varepsilon_3)\{ (q/a)[(\varepsilon_3/\varepsilon_2) - 1] - (q/b)[(\varepsilon_3/\varepsilon_2) - (\varepsilon_3/\varepsilon_1)] \}$$

$$= (1/4\pi)\{ (q/a)[1/\varepsilon_2 - 1/\varepsilon_3] - (q/b)[1/\varepsilon_2 - 1/\varepsilon_1] \}$$

So here are all the constants

$$A = 0$$
$$B = (1/4\pi\varepsilon_3)\{ (q/a)[(\varepsilon_3/\varepsilon_2) - 1] - (q/b)[(\varepsilon_3/\varepsilon_2) - (\varepsilon_3/\varepsilon_1)] \}$$
$$C = (1/4\pi\varepsilon_3)(\varepsilon_3/\varepsilon_2)$$
$$D = 0$$
$$E = -(q/b)(1/4\pi\varepsilon_3)[(\varepsilon_3/\varepsilon_2) - (\varepsilon_3/\varepsilon_1)]$$
$$F = +(1/4\pi\varepsilon_3)(\varepsilon_3/\varepsilon_1)$$

Now define

$$\kappa = (1/4\pi\varepsilon_3)$$

Then we have

$$A = 0$$
$$B = \kappa\{ (q/a)[(\varepsilon_3/\varepsilon_2) - 1] - (q/b)[(\varepsilon_3/\varepsilon_2) - (\varepsilon_3/\varepsilon_1)] \} = \kappa (q/a)[(\varepsilon_3/\varepsilon_2) - 1] + E$$
$$C = \kappa(\varepsilon_3/\varepsilon_2)$$
$$D = 0$$
$$E = -\kappa (q/b)[(\varepsilon_3/\varepsilon_2) - (\varepsilon_3/\varepsilon_1)]$$
$$F = +\kappa(\varepsilon_3/\varepsilon_1)$$

So write the non-trivial ones again

$C = \kappa\,(\varepsilon_3/\varepsilon_2)$
$F = \kappa\,(\varepsilon_3/\varepsilon_1)$
$E = -\kappa\,(q/b)\,[\,(\varepsilon_3/\varepsilon_2) - (\varepsilon_3/\varepsilon_1)]$
$B = \kappa\,(q/a)\,[(\varepsilon_3/\varepsilon_2) - 1] + E$

$c = (\varepsilon_3/\varepsilon_2)$	$\kappa c = (1/4\pi\varepsilon_2)$
$f = (\varepsilon_3/\varepsilon_1)$	$\kappa f = (1/4\pi\varepsilon_1)$
$e = -(q/b)\,[\,(\varepsilon_3/\varepsilon_2) - (\varepsilon_3/\varepsilon_1)]$	$\kappa e = -(q/b)\,(1/4\pi)\,[1/\varepsilon_2 - 1/\varepsilon_1]$
$\beta = (q/a)\,[(\varepsilon_3/\varepsilon_2) - 1] + e$	$\kappa\beta = (q/a)\,(1/4\pi)\,[1/\varepsilon_2 - 1/\varepsilon_3] + \kappa e$

The potentials and fields are then

$E_3 = \kappa\, q/r^2$	$= (1/4\pi\varepsilon_3)\,(q/r^2)$
$E_2 = \kappa c\, q/r^2$	$= (1/4\pi\varepsilon_2)\,(q/r^2)$
$E_1 = \kappa f\, q/r^2$	$= (1/4\pi\varepsilon_1)\,(q/r^2)$

$\varphi_3 = \kappa\, q/r + \kappa\beta$	$= (1/4\pi\varepsilon_3)\,(q/r) + (q/a)\,(1/4\pi)\,[1/\varepsilon_2 - 1/\varepsilon_3] + \kappa e$
$\varphi_2 = \kappa c\, q/r + \kappa e$	$= (1/4\pi\varepsilon_2)\,(q/r) - (q/b)\,(1/4\pi)\,[1/\varepsilon_2 - 1/\varepsilon_1]$
$\varphi_1 = \kappa f\, q/r$	$= (1/4\pi\varepsilon_1)\,(q/r)$

In all three regions, the field goes as constant/r^2 as you might guess.

Fact: In each region, the E field thinks all space is filled with *its* ε. That is a simple and fascinating result. You can see this would be true with any number of spherical layers.

In Maple I enter all the stuff and then check BC's: (Here $\kappa = (1/4\pi\varepsilon_3) = 1$)

```
c := e3/e2:
f := e3/e1:
e := -(q/b)*((e3/e2)-(e3/e1)):
beta := (q/a)*((e3/e2)-1) + e:
phi3 := (r) -> k*(q/r + beta):
phi2 := (r) -> k* (c*q/r + e):
phi1 := (r) -> k*f*q/r:

phi1(b)-phi2(b): simplify(%);
```

$$0$$

```
phi2(a)-phi3(a): simplify(%);
```

$$0$$

```
E3 := (r) -> k*q/r^2:
E2 := (r) -> k*c*q/r^2:
E1 := (r) -> k*f*q/r^2:
e1*E1(b)-e2*E2(b);
```

$$0$$

```
e3*E3(a)-e2*E2(a);
```

$$0$$

and all is well. I then generate the complete φ and E functions and set in some constants

```
phi := piecewise(r>=0 and r<a,phi3,r>=a and r<b,phi2,r>=b,phi1);
```

$$\phi := \begin{cases} \phi 3 & -r \le 0 \text{ and } r - a < 0 \\ \phi 2 & a - r \le 0 \text{ and } r - b < 0 \\ \phi 1 & b \le r \end{cases}$$

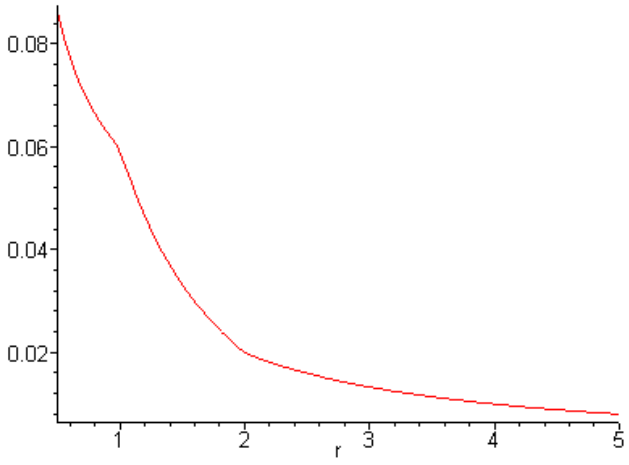
```
E := piecewise(r>=0 and r<a,E3,r>=a and r<b,E2,r>=b,E1);
```

$$E := \begin{cases} E3 & -r \le 0 \text{ and } r - a < 0 \\ E2 & a - r \le 0 \text{ and } r - b < 0 \\ E1 & b \le r \end{cases}$$

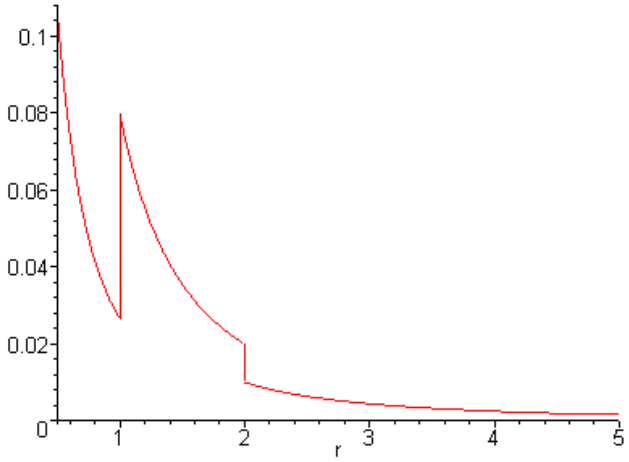
```
k := 1/ (4*Pi*e3):
e1 := 2: e2 := 1: e3 := 3: q := 1: a := 1: b := 2:
```

And finally here are some plots

```
plot(phi(r),r=0.5..5);
```




```
plot(E(r),r=0.5..5);
```



And here you see the expected two kinks in φ and the expected two jumps in E.

Now what is the charge Q mentioned earlier?

Start with

div **E** = ρ_{pol}/ε_0 . // since there is no free charge in this problem other than q.

Now use area pointing radially outward for d**A** and then use

$(1/\varepsilon_0)\int_V \rho_{pol}\, dV = \int_S \mathbf{E} \bullet d\mathbf{A}$

Apply this with a pillbox across the 1,2 boundary to get (r = b)

$(1/\varepsilon_0)n_{pol}^{(12)} = (E_1 - E_2) = \kappa f\, q/b^2 - \kappa c\, q/b^2$

$= \kappa\,(q/b^2)(f\text{-}c) = (1/4\pi\varepsilon_3)\,(q/b^2)\;[(\varepsilon_3/\varepsilon_1)\;-(\varepsilon_3/\varepsilon_2)] = (1/4\pi)\,(q/b^2)\;[(1/\varepsilon_1)\;-(1/\varepsilon_2)]$

and therefore

$n_{pol}^{(12)} = (1/4\pi)\,[(\varepsilon_0/\varepsilon_1)\;-(\varepsilon_0/\varepsilon_2)]\,(q/b^2)$

The total charge on this boundary is then

$Q_{12} = 4\pi b^2 * n_{pol}^{(12)} = [(\varepsilon_0/\varepsilon_1)\;-(\varepsilon_0/\varepsilon_2)]\,q$.

Next, put a pillbox across the 2,3 boundary to get

$(1/\varepsilon_0)n_{pol}^{(23)} = (E_2\;-E_3) = \kappa c\;q/a^2 - \kappa\;q/a^2$

$= \kappa\,(q/a^2)[\,(\varepsilon_3/\varepsilon_2) - 1] = (1/4\pi\varepsilon_3)\,(q/a^2)[\,(\varepsilon_3/\varepsilon_2) - 1] = (1/4\pi)\,(q/a^2)[\,(1/\varepsilon_2) - (1/\varepsilon_3)\,]$

so that

$n_{pol}^{(23)} = (1/4\pi)\,(q/a^2)\,[\,(\varepsilon_0/\varepsilon_2) - (\varepsilon_0/\varepsilon_3)\,]$

and then

$Q_{23} = 4\pi a^2 * n_{pol}^{(23)} = q\,[\,(\varepsilon_0/\varepsilon_2) - (\varepsilon_0/\varepsilon_3)\,]$

So here are the polarization charge results:

$n_{pol}^{(12)} = (1/4\pi)\,[(\varepsilon_0/\varepsilon_1)\;-(\varepsilon_0/\varepsilon_2)]\,(q/b^2)$
$n_{pol}^{(23)} = (1/4\pi)\,[\,(\varepsilon_0/\varepsilon_2) - (\varepsilon_0/\varepsilon_3)\,]\,(q/a^2)$

$Q_{12} = [(\varepsilon_0/\varepsilon_1)\;-(\varepsilon_0/\varepsilon_2)]\,q = Q_1 - Q_2$ (*)
$Q_{23} = [(\varepsilon_0/\varepsilon_2)\;-(\varepsilon_0/\varepsilon_3)]\,q = Q_2 - Q_3$ (**)

Now in region 3 we have $E = (1/4\pi\varepsilon_3)\,q/r = (1/4\pi\varepsilon_0)\,[(\varepsilon_0/\varepsilon_3)q]/r$ so E there is seeing a total vacuum charge of $[(\varepsilon_0/\varepsilon_3)q]$. But this must be $q + Q_3$ which tells us that

$Q_3 = (\varepsilon_0/\varepsilon_3)q - q = [(\varepsilon_0/\varepsilon_3) - 1\,]\,q$ which we see is negative as expected (as Q_3 defined)

Then from (**) we get

$Q_2 = Q_{23} + Q_3 = [(\varepsilon_0/\varepsilon_2)\;-(\varepsilon_0/\varepsilon_3)]\,q + [(\varepsilon_0/\varepsilon_3) - 1\,]\,q = [(\varepsilon_0/\varepsilon_2) - 1\,]\,q < 0$

and then from (*) we get

$Q_1 = Q_{12} + Q_2 = [(\varepsilon_0/\varepsilon_1)\;-(\varepsilon_0/\varepsilon_2)]\,q + [(\varepsilon_0/\varepsilon_2) - 1\,]\,q = [(\varepsilon_0/\varepsilon_1) - 1\,]\,q < 0$

So here are the complete charge results:

$$n_{pol}^{(12)} = (1/4\pi)\,[(\varepsilon_0/\varepsilon_1) - (\varepsilon_0/\varepsilon_2)]\,(q/b^2)$$
$$n_{pol}^{(23)} = (1/4\pi)\,[(\varepsilon_0/\varepsilon_2) - (\varepsilon_0/\varepsilon_3)]\,(q/a^2)$$

$$Q_{12} = [(\varepsilon_0/\varepsilon_1) - (\varepsilon_0/\varepsilon_2)]\,q = Q_1 - Q_2 \qquad (*)$$
$$Q_{23} = [(\varepsilon_0/\varepsilon_2) - (\varepsilon_0/\varepsilon_3)]\,q = Q_2 - Q_3 \qquad (**)$$

$$Q_1 = [(\varepsilon_0/\varepsilon_1) - 1]\,q < 0$$
$$Q_2 = [(\varepsilon_0/\varepsilon_2) - 1]\,q < 0$$
$$Q_3 = [(\varepsilon_0/\varepsilon_3) - 1]\,q < 0$$

Here is an alternative way to determine the charges.

If you are in region 3, you have $E = (1/4\pi\varepsilon_3)(q/r^2) = (1/4\pi\varepsilon_0)([(\varepsilon_0/\varepsilon_3)q]/r^2)$, so region 3 is seeing a total effective vacuum point charge of $(\varepsilon_0/\varepsilon_3)q$. We can equate this to $q + Q_3$ if we add up the one pol charge that observer sees from region 3. Thus, $q + Q_3 = (\varepsilon_0/\varepsilon_3)q$ and so

$$Q_3 = [(\varepsilon_0/\varepsilon_3) - 1]\,q < 0$$

If you are in region 2, you have $E = (1/4\pi\varepsilon_2)(q/r^2) = (1/4\pi\varepsilon_0)([(\varepsilon_0/\varepsilon_2)q]/r^2)$, so region 2 is seeing a total effective vacuum point charge of $(\varepsilon_0/\varepsilon_2)q$. We can equate this to $q + Q_3 - Q_3 + Q_2$ if we include the three pol charges that observer sees from region 2. Thus, $q + Q_2 = (\varepsilon_0/\varepsilon_2)q$ and so

$$Q_2 = [(\varepsilon_0/\varepsilon_2) - 1]\,q < 0$$

If you are in region 1, you have $E = (1/4\pi\varepsilon_1)(q/r^2) = (1/4\pi\varepsilon_0)([(\varepsilon_0/\varepsilon_1)q]/r^2)$, so region 1 is seeing a total effective vacuum point charge of $(\varepsilon_0/\varepsilon_1)q$. We can equate this to $q + Q_3 - Q_3 + Q_2_Q_2 + Q_1$ if we include the five pol charges that observer sees from region 1. Thus, $q + Q_1 = (\varepsilon_0/\varepsilon_1)q$ and so

$$Q_1 = [(\varepsilon_0/\varepsilon_1) - 1]\,q$$

Thus we replicate the earlier Q_i results.