Point charge in two dielectrics, Part 2 PhL 10.24.13

There is more stuff I need to know about this problem before I can attack my two King logjam question.

First of all, since div **D** = ρ = the free charge only, you would think that the field **D** would be

$$\mathbf{D_1} = \mathbf{D_2} = q\,(1/4\pi)\,(1/r)\,\hat{\mathbf{r}}$$

since **D** does not see the polarization charge at the boundary. Here we place the Jackson region 1 point charge at a spherical system origin.

We then know that

$\mathbf{P_1} = \varepsilon_0\chi_1\,\mathbf{E_1}$ in region 1
$\mathbf{P_2} = \varepsilon_0\chi_2\,\mathbf{E_2}$ in region 2

$\mathbf{D_1} = \varepsilon_1\mathbf{E_1}$ in region 1
$\mathbf{D_2} = \varepsilon_2\mathbf{E_2}$ in region 2

But since $\mathbf{D_1} = \mathbf{D_2} = \mathbf{D}$ = the same in both regions, we seem to get

$q\,(1/4\pi)\,(1/r)\,\hat{\mathbf{r}} = \varepsilon_1\mathbf{E_1}$ in region 1
$q\,(1/4\pi)\,(1/r)\,\hat{\mathbf{r}} = \varepsilon_2\mathbf{E_2}$ in region 2

But this is NOT what we get. The reason is that in region 1 we don't get $\mathbf{E_1} = [...]\hat{\mathbf{r}}$, but instead we know the E field lines are curved. So what have I done wrong here?

What is wrong is this. Knowing that div **D** = ρ does not determine **D**. This is just a condition on D. The other major condition is curl **E** = **0** for a static situation. And we have to make sure things are OK at boundaries, and THEN we have a solution.

I said that **D** does not see polarization charge at a boundary. That is true, but curl **E** = **0** does see polarization charge at a boundary. Recall the integral form of our two rules,

$$\text{div}\,\mathbf{D} = \rho \quad \Leftrightarrow \quad \int_V \rho\,dV = \int_S \mathbf{D}\bullet d\mathbf{A} \qquad (1.1.13)$$

$$\text{curl}\,\mathbf{E} = 0 \quad \Leftrightarrow \quad \oint \mathbf{E}\bullet d\mathbf{s} = 0 \qquad (1.1.17)$$

This means that $E_{||}$ is the same on both sides of a boundary, while $D_\perp$ is the same on both sides. For our little Jackson problem that means

$E_{||}^{(1)} = E_{||}^{(2)}$
$\varepsilon_1 E_\perp^{(1)} = \varepsilon_2 E_\perp^{(2)}$

or

$(1/\varepsilon_1)D_{||}^{(1)} = (1/\varepsilon_2)D_{||}^{(2)}$

$D_{\perp}^{(1)} = D_{\perp}^{(2)}$

Thus, even thought D does not see polarization charge at a boundary, it does not simply "pass through" a boundary as I suggested above. There is refraction of D at the boundary just as with E !

Now how would you approach solving the D problem? I don't really know! I only know how to approach the φ problem and $\mathbf{E} = -\nabla\varphi$ so I am better at doing the E problem.

My "cylindrical atoms" solution to the problem was this:

$$\varphi_1(\rho,z) = (1/4\pi\varepsilon_1) \int_0^\infty dk\, e^{-k|z|} J_0(k\rho) + \int_0^\infty dk\, A_k\, e^{kz} J_0(k\rho)$$

$$\varphi_2(\rho,z) = \int_0^\infty dk\, B_k\, e^{-kz} J_0(k\rho) \qquad R = \sqrt{\rho^2 + s^2}$$

$$E_{z1} = \partial_z\varphi_1 = (1/4\pi\varepsilon_1) \int_0^\infty dk\, (\partial_z e^{-k|z|}) J_0(k\rho) + \int_0^\infty dk\, A_k\, k\, e^{kz} J_0(k\rho)$$

$$E_{z2} = \partial_z\varphi_2 = \int_0^\infty dk\, B_k\, (-k)\, e^{-kz} J_0(k\rho)$$

$$E_{\rho 1} = \partial_\rho\varphi_1 = (1/4\pi\varepsilon_1) \int_0^\infty dk\, k\, e^{-k|z|} J_0'(k\rho) + \int_0^\infty dk\, kA_k\, e^{kz} J_0'(k\rho)$$

$$E_{\rho 2} = \partial_\rho\varphi_2 = \int_0^\infty dk\, kB_k\, e^{-kz} J_0'(k\rho)$$

$$A_k = (1/4\pi\varepsilon_1)e^{-2ks}\,[(\varepsilon_1-\varepsilon_2)/(\varepsilon_1+\varepsilon_2)]$$

$$B_k = (1/4\pi\varepsilon_1)\,[2\varepsilon_1/(\varepsilon_1+\varepsilon_2)] \equiv B$$

For fun, let's look at the ratio of $E_{z2}/E_{\rho 2}$

$$E_{z2}/E_{\rho 2} = \int_0^\infty dk\, B_k\, (-k)\, e^{-kz} J_0(k\rho) \;/\; \int_0^\infty dk\, k\, B_k\, e^{-kz} J_0'(k\rho)$$

$$= -\int_0^\infty dk\, k\, e^{-kz} J_0(k\rho) \;/\; \int_0^\infty dk\, k\, e^{-kz} J_0'(k\rho)$$

where the B_k just cancel, being constants as shown above. If we use $\rho^2 + z^2 = r^2$ to eliminate either ρ or z, the solution claims that this ratio is a constant independent of r ! That result seems completely non-obvious looking at the integrals. Maybe write this ratio this way

$$E_{z2}/E_{\rho 2} = \partial_z\varphi_2 \;/\; \partial_\rho\varphi_2 \qquad \text{where } \varphi_2 = B \int_0^\infty dk\, e^{-kz} J_0(k\rho)$$

For $z > 0$, we can see that $\int_0^\infty dk\, e^{-kz}\, J_0(k\rho) = 1/R$ for a unit point charge at the origin, but this is true only for $z > 0$!! That is why we get the lines there all zeroing in on the location of q.

So it is quite easy to obtain the image charge solution from my results above.

What about the surface charge distribution? We get that from

$$\text{div } \mathbf{D} = \rho \qquad \Leftrightarrow \qquad \int_V \rho\, dV = \int_S \mathbf{D} \bullet d\mathbf{A} \tag{1.1.13}$$

If we apply this to a tiny pillbox straddling the boundary, we find that

$$n\, dA = D_{z2} dA - D_{z1} dA \quad \Rightarrow \quad n = D_{z2} - D_{z1}$$

$$\Rightarrow \; n = \varepsilon_2 E_{z2} - \varepsilon_1 E_{z1}$$

where from above, at $z = s$,

$$E_{z1} = \partial_z \varphi_1 = (1/4\pi\varepsilon_1) \int_0^\infty dk\, (-k e^{-ks}) J_0(k\rho) + \int_0^\infty dk\, A_k\, k\, e^{ks}\, J_0(k\rho)$$

$$E_{z2} = \partial_z \varphi_2 = B \int_0^\infty dk\, (-k)\, e^{-ks}\, J_0(k\rho)$$

$$A_k = (1/4\pi\varepsilon_1) e^{-2ks} [(\varepsilon_1-\varepsilon_2)/(\varepsilon_1+\varepsilon_2)] = (1/4\pi\varepsilon_1) e^{-2ks}\, f \qquad f = [(\varepsilon_1-\varepsilon_2)/(\varepsilon_1+\varepsilon_2)]$$
$$B = (1/4\pi\varepsilon_1)\, [2\varepsilon_1 / (\varepsilon_1+\varepsilon_2)] = (1/4\pi\varepsilon_1)\, g \qquad g = [2\varepsilon_1 / (\varepsilon_1+\varepsilon_2)]$$

Therefore

$$E_{z1} = \partial_z \varphi_1 = -(1/4\pi\varepsilon_1) \int_0^\infty dk\, k\, e^{-ks} J_0(k\rho) + (1/4\pi\varepsilon_1)\, f \int_0^\infty dk\, k\, e^{-ks}\, J_0(k\rho)$$

$$= (1/4\pi\varepsilon_1)\, [\, f - 1\,] \int_0^\infty dk\, k\, e^{-ks}\, J_0(k\rho)$$

$$E_{z2} = (1/4\pi\varepsilon_1)\, [-g] \int_0^\infty dk\; k\, e^{-ks}\, J_0(k\rho)$$

Then the surface charge density seems to be this

$$n = \varepsilon_2 E_{z2} - \varepsilon_1 E_{z1}$$
$$= (1/4\pi\varepsilon_1)\, [-\varepsilon_2 g - \varepsilon_1 (f-1)] \int_0^\infty dk\, k\; e^{-ks}\, J_0(k\rho)$$
$$= -(1/4\pi\varepsilon_1)\, [\varepsilon_2 g + \varepsilon_1 (f-1)] \int_0^\infty dk\, k\; e^{-ks}\, J_0(k\rho)$$

Meanwhile, Maple tells us that $[\varepsilon_2 g + \varepsilon_1(f-1)] = 0$ so our result is that n = 0. This is the correct result for the free charge at the boundary!

```
f := (e1-e2)/(e1+e2);
```

$$f := \frac{e1 - e2}{e1 + e2}$$

```
g := 2*e1/(e1+e2);
```

$$g := 2\,\frac{e1}{e1 + e2}$$

```
t := e2*g + e1*(f-1):simplify(%);
```

$$0$$

I think the integral shown is a simple elementary function but hold on that for a moment.

Now let's find the charge that E sees!

$$\text{div } \mathbf{E} = \rho/\varepsilon \quad \Leftrightarrow \quad \int_V (\rho/\varepsilon)\, dV = \int_S \mathbf{E} \bullet d\mathbf{A} \quad [****] \qquad (1.1.14)$$

If we apply this to a tiny pillbox straddling the boundary, what do we do about ε? Suppose we put half the charge into each half space, just as something to try. Then we use

we find that

$$n\, dA = \varepsilon_2 E_{z2} dA - D_{z1} dA \quad \Rightarrow \quad n = D_{z2} - D_{z1}$$

$$\Rightarrow \quad n = \varepsilon_2 E_{z2} - \varepsilon_1 E_{z1}$$