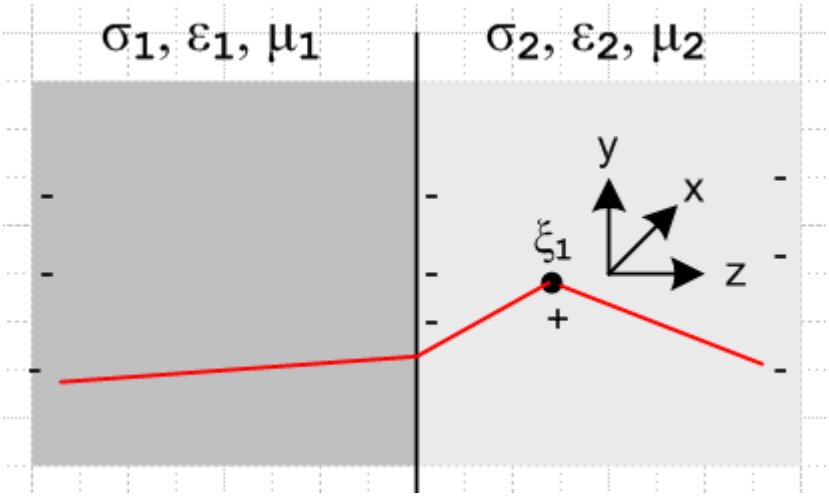
A problem in electrostatics PhL 10.22.13

See red comment in later versions.

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1. Preliminaries

Here is the picture, and I admit that I don't even know what to expect here; I draw two tentative electric field lines. I put the test point charge in region 2.



I think each region has its own Poisson equation

$\nabla^2\varphi_1 = -\delta(x-\xi_1)$ $\quad \varphi_1(-\infty) = 0$

$\nabla^2\varphi_2 = 0$ $\quad \varphi_2(+\infty) = 0$

$\varphi_1(z=0) = \varphi_2(z=0)$

So I want the potential to be continuous at the boundary (I think). I also know at the boundary that

$$\varepsilon_2 E_{n2} = \varepsilon_1 E_{n1}$$
$$E_{||,2} = E_{||,1}$$

Recalling that

$$n = \varepsilon_2 E_{z2} - \varepsilon_1 E_{z1}$$

we conclude that there is no surface charge n at the boundary in this problem!

I think from distant memory that the solution of this problem is the in region 1 the potential is just that of ξ_1 while in region 2 it is that of the point charge moved to some image position as my lines suggest. BUT, Bleaney page 21 has a picture which claims the E lines are curved in region 1 and come in normal to the boundary. So I guess I need to try to solve this problem exactly and see what goes on.

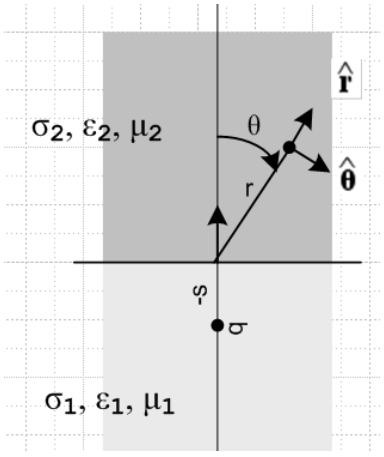
Notice that in region 1 we don't really have a Green's function problem because we don't a priori know that $\varphi = 0$ on the boundary, and probably $\varphi \neq 0$ there.

2. Try the Smythian method in spherical coordinates: Attempt #1

I might try the Smythian method here. The 3D harmonics or atoms are these,

$$[Ar^n + Br^{-n-1}]\, Y_{nm}(\theta,\varphi)$$

Let's put the coordinate system origin on the boundary at the level of ξ_1. In fact, let's just restate the problem from scratch like so:



Since the solution will have azimuthal symmetry, our atoms simplify to

$$\varphi_1(r,\theta) = \Sigma_{n=0}^{\infty}[A_n r^n + B_n r^{-n-1}]\, P_n(\cos\theta)$$
$$\varphi_2(r,\theta) = \Sigma_{n=0}^{\infty}[C_n r^n + D_n r^{-n-1}]\, P_n(\cos\theta)$$

At a point on the boundary we have $\theta = \pi/2$ so

$$\varphi_1(r,\pi/2) = \Sigma_{n=0}^{\infty}[A_n r^n + B_n r^{-n-1}]\, P_n(0)$$
$$\varphi_2(r,\pi/2) = \Sigma_{n=0}^{\infty}[C_n r^n + D_n r^{-n-1}]\, P_n(0)$$

which we let rest for a moment. Now

$$\mathbf{E}_1(r,\theta) = -\nabla\varphi_1 = \partial_r\varphi_1(r,\theta)\hat{\mathbf{r}} + (1/r)\partial_\theta\varphi_1(r,\theta)\hat{\boldsymbol{\theta}} = E_{r1}(r,\theta)\ \hat{\mathbf{r}} + E_{\theta 1}(r,\theta)\ \hat{\boldsymbol{\theta}}$$
$$\mathbf{E}_2(r,\theta) = -\nabla\varphi_2 = \partial_r\varphi_2(r,\theta)\hat{\mathbf{r}} + (1/r)\partial_\theta\varphi_2(r,\theta)\hat{\boldsymbol{\theta}} = E_{r2}(r,\theta)\ \hat{\mathbf{r}} + E_{\theta 2}(r,\theta)\ \hat{\boldsymbol{\theta}}$$

We can calculate

$$\partial_\theta P_n(\cos\theta) = d/d\theta\ P_n(\cos\theta) = dz/d\theta * d/dz\ P_n(z) = -\sin\theta\ P_n'(z)$$

and then

$$\partial_r\varphi_1(r,\theta) = \Sigma_{n=0}^{\infty}[nA_n r^{n-1} - (n+1)\, B_n r^{-n-2}]\, P_n(z) = E_{r1}(r,\theta)$$
$$\partial_\theta\varphi_1(r,\theta) = -\sin\theta\ \Sigma_{n=0}^{\infty}[A_n r^n + B_n r^{-n-1}]\, P_n'(z) = r\, E_{\theta 1}(r,\theta)$$

$$\partial_r\varphi_2(r,\theta) = \Sigma_{n=0}^{\infty}[nC_n r^{n-1} - (n+1)\, D_n r^{-n-2}]\, P_n(z) = E_{r2}(r,\theta)$$
$$\partial_\theta\varphi_2(r,\theta) = -\sin\theta\ \Sigma_{n=0}^{\infty}[C_n r^n + D_n r^{-n-1}]\, P_n'(z) = r\, E_{\theta 2}(r,\theta)$$

If we now go to the boundary which is $\theta = \pi/2$ we find that $\hat{\mathbf{r}}$ is a parallel unit vector and $\hat{\boldsymbol{\theta}}$ is normal, and recall

$$\varepsilon_2\, E_{n2} = \varepsilon_1\, E_{n1}$$
$$E_{||,2} = E_{||,1}$$

which says

$$\varepsilon_2\ E_{\theta 2}(r,\pi/2) = \varepsilon_1 E_{\theta 1}(r,\pi/2)$$
$$E_{r2}(r,\pi/2) = E_{r1}(r,\pi/2)$$

and with the expansions we then have

$$\varepsilon_2\ \{ -1\ \Sigma_{n=0}^{\infty}[C_n r^n + D_n r^{-n-1}]\, P_n'(\pi/2)\ \} = \varepsilon_1\ \{ -1\ \Sigma_{n=0}^{\infty}[A_n r^n + B_n r^{-n-1}]\, P_n'(0)\ \}$$

$$\Sigma_{n=0}^{\infty}[nC_n r^{n-1} - (n+1)\, D_n r^{-n-2}]\, P_n(\pi/2) = \Sigma_{n=0}^{\infty}[nA_n r^{n-1} - (n+1)\, B_n r^{-n-2}]\, P_n(0)$$

But we now go look up to find that

$$P_n^m(x=0) = (-1)^{(n+m)/2}\,(n+m-1)!!/\,(n-m)!! \qquad \text{// } n+m = \text{even}$$
$$P_n^m(x=0) = 0 \qquad \text{// } n+m = \text{odd}$$

so then

$P_n(x=0) = (-1)^{(n)/2} (n-1)!!/ (n)!!$ // n = even

$P_n(x=0) = 0$ // n = odd

Thus our expansions are really this

$$\varepsilon_2 \{ - 1 \Sigma_{n,even}[C_n r^n + D_n r^{-n-1}] P_n'(\pi/2) \} = \varepsilon_1 \{ - 1 \Sigma_{n,even} [A_n r^n + B_n r^{-n-1}] P_n'(0) \}$$

$$\Sigma_{n,even} [nC_n r^{n-1} - (n+1) D_n r^{-n-2}] P_n(\pi/2) = \Sigma_{n,even} [nA_n r^{n-1} - (n+1) B_n r^{-n-2}] P_n(0)$$

We learn nothing at all here about the possible odd n coefficients.

In each line, I think we have to balance the powers of r to get an equality, so then

$\varepsilon_2 C_n = \varepsilon_1 A_n$ 1

$\varepsilon_2 D_n = \varepsilon_1 B_n$ 2

$nC_n = nA_n$ 3

$D_n = B_n$ 4 n = even

Assuming $\varepsilon_1 \neq \varepsilon_2$, from 2 and 4 we find that we must have $D_n = 0$ and $B_n = 0$ for n = even, which is a great simplification. We really knew that from the start since we could not have blow up at the origin, but here we got the result a different way. Then we are left with these two BC's

$\varepsilon_2 C_n = \varepsilon_1 A_n$ 1

$nC_n = nA_n$ 3 n = even

If $n \neq 0$, this says $A_n = 0$ and $C_n = 0$, but for n = 0 we are OK.

So here is what we have found out:

For n = even > 0: $D_n = 0$ and $B_n = 0$

$A_n = 0$ and $C_n = 0$

For n = 0: $D_n = 0$ and $B_n = 0$

Thus we can write our Smythian expansions in this manner

$$\varphi_1(r,\theta) = A_0 + \Sigma_{n=odd} A_n r^n P_n(\cos\theta)$$

$$\varphi_2(r,\theta) = C_0 + \Sigma_{n=odd} C_n r^n P_n(\cos\theta)$$

We have fully incorporated the electric field BC's at the boundary.

Now what about the continuity of φ at the boundary?

$$\varphi_1(r,\pi/2) = A_0 + \Sigma_{n=odd}\ A_n r^n P_n(0)$$

$$\varphi_2(r,\pi/2) = C_0 + \Sigma_{n=odd}\ C_n r^n P_n(0)$$

But $P_n(0) = 0$ for n odd, so we really have

$$\varphi_1(r,\pi/2) = A_0$$

$$\varphi_2(r,\pi/2) = C_0$$

which tells us that

$$A_0 = C_0$$

and then our potentials must have these Smythian forms

$$\varphi_1(r,\theta) = A_0 + \Sigma_{n=odd}\ A_n r^n P_n(\cos\theta)$$

$$\varphi_2(r,\theta) = A_0 + \Sigma_{n=odd}\ C_n r^n P_n(\cos\theta)$$

The corresponding fields are

$$E_{r1}(r,\theta) = \partial_r\varphi_1(r,\theta) = \Sigma_{n=odd}\ nA_n r^{n-1} P_n(\cos\theta)$$

$$E_{\theta 1}(r,\theta) = 1/r * \partial_\theta\varphi_1(r,\theta) = -\sin\theta\,(1/r)\,\Sigma_{n=odd}\ A_n r^n P_n(z) = -\sin\theta\,\Sigma_{n=odd}\ A_n r^{n-1} P_n(z)$$

and similarly in region 2. It seems unreasonable to have $n > 1$ because if so, the fields blow up at infinity! So I think we are now limited to $n = 1$ as the only term and we have

$$\varphi_1(r,\theta) = A_0 + A_1 r\cos\theta$$
$$\varphi_2(r,\theta) = A_0 + C_1 r\cos\theta$$

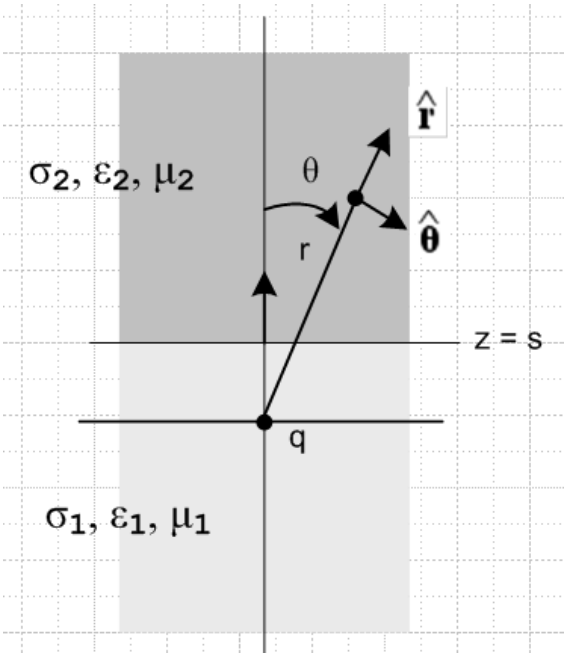
$$E_{r1}(r,\theta) = A_1\cos\theta \qquad E_{\theta 1}(r,\theta) = -\sin\theta A_1\cos\theta$$
$$E_{r2}(r,\theta) = C_1\cos\theta \qquad E_{\theta 1}(r,\theta) = -\sin\theta C_1\cos\theta$$

Why am I allowing the potential to blow up?

Put a Halt on this attempt and start over.

3. Try the Smythian method in spherical coordinates: Attempt #2

STOP! I made a bad choice of coordinate system? The potential should blow up at the point charge, but my solution does not show that boundary condition. Try this coordinate system:



Then start all over with the general atomic forms

$$\varphi_1(r,\theta) = \Sigma_{n=0}^{\infty}[A_n r^n + B_n r^{-n-1}]\, P_n(\cos\theta)$$
$$\varphi_2(r,\theta) = \Sigma_{n=0}^{\infty}[C_n r^n + D_n r^{-n-1}]\, P_n(\cos\theta)$$

Now right off the bat, if we go far away we expect potential to look somehow like 1/r so that means we have to have $A_n = C_n = 0$ for ALL n. But there could be powers like $1/r^2$ somehow created by the boundary, so don't throw them out. It just seems you cannot have higher powers like r^0 or r^1 and so on. So write

$$\varphi_1(r,\theta) = \Sigma_{n=0}^{\infty} B_n r^{-n-1} P_n(\cos\theta)$$
$$\varphi_2(r,\theta) = \Sigma_{n=0}^{\infty} D_n r^{-n-1} P_n(\cos\theta)$$

Now try to match these on the boundary where $r\cos\theta = s$. Not so easy to do this.

$$\varphi_1(s/\cos\theta,\theta) = \Sigma_{n=0}^{\infty} B_n (s/\cos\theta)^{-n-1} P_n(\cos\theta)$$
$$\varphi_2(s/\cos\theta,\theta) = \Sigma_{n=0}^{\infty} D_n (s/\cos\theta)^{-n-1} P_n(\cos\theta)$$

This seems to force $B_n = D_n$ and then we have the same potential everywhere, which is no good.

And it suggests (at least) that maybe the only surviving coefficients are in fact B_0 and D_0 in order to get 1/r . Then we just have

$$\varphi_1(r,\theta) = B_0 r^{-1}$$
$$\varphi_2(r,\theta) = D_0 r^{-1}$$

which is pretty simple. In our new coordinate system the boundary is located at z = s which is rcosθ = s. So maybe write this as r = s/cosθ. Then we have, on the boundary,

s $\varphi_1(s/\cos\theta,\theta) = B_0\cos\theta/s$
$\varphi_2(s/\cos\theta,\theta) = D_0\cos\theta/s$

That seems to suggest that $B_0 = D_0$ and we now have this universal potential

$\varphi_1(r,\theta) = B_0 r^{-1}$
$\varphi_2(r,\theta) = B_0 r^{-1}$

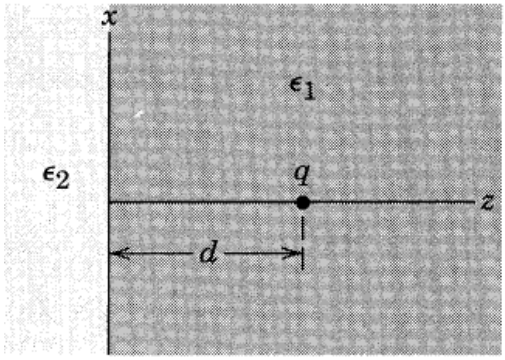
But this implies that the boundary has no affect whatsoever, and that cannot possibly be right and certainly does not agree with the Bleaney picture.

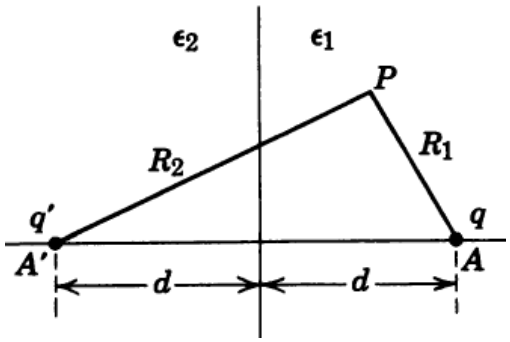
Put a Halt on this attempt and start over.

4. Jackson's Solution

STOP. I have flailed enough this morning. It is time to go look up how this elementary problem is done, I have a mental block against it, I have some bad assumption. Maybe φ is not continuous? So who is going to give me the solution? I just have to start perusing books or the web.

Jacksons? This is too simple a problem for Jackson. But blue Jackson draws the Bleaney picture! Jackson's entire Section 4.4 is a treatment of this problem! He has the charge at z = +d. He puts the boundary at z = 0 as in my original picture. He is going to use the "method of images". Here are his pictures:





He puts an image charge q' at z = -d, just an ansatz. Here is the potential in region 1 in SI units,

$$\Phi = \frac{1}{4\pi\epsilon_1}\left(\frac{q}{R_1} + \frac{q'}{R_2}\right), \qquad z > 0 \qquad (4.43)$$

where $R_1 = \sqrt{\rho^2 + (d - z)^2}$, $R_2 = \sqrt{\rho^2 + (d + z)^2}$. So far the procedure is region 1

He uses cylindrical coordinates, I was using sphericals.

Now for region 2 he tries this ansatz,

$$\Phi = \frac{1}{4\pi\epsilon_2}\frac{q''}{R_1}, \qquad z < 0 \qquad (4.44)$$

where some image is put at exactly z = d again! We now have q' and q" both unknown.

He then computes the fields in regions 1 and 2 from the assumed potential forms, and then he applies the E field boundary conditions (not a φ one!) and he concludes that the problem is solved if you choose

$$q' = -\left(\frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1}\right)q$$
$$q'' = \left(\frac{2\epsilon_2}{\epsilon_2 + \epsilon_1}\right)q \qquad (4.45)$$

This gives a solution because it satisfies all the BC's at the boundary, and it also has 1/r at large r. So here is his solution

$\varphi = (1/4\pi\varepsilon_1)\,(\ q/R_1 + q'/R_2)$ $\qquad z > 0$

$\varphi = (1/4\pi\varepsilon_2)\,(\ q''/R_1\ )$ $\qquad z < 0$

Jackson then calculates the polarization charge density on the boundary.

On the boundary z = 0 we have $R_1 = R_2 \equiv R$ so we have

$\varphi_1 = (1/4\pi\varepsilon_1)\,(\ q + q')\ /R$

$\varphi_2 = (1/4\pi\varepsilon_2)\,(\ q''\ /R\ )$

Maple shows that these do in fact match at the boundary,

```
qp := - q*(e2-e1)/(e2+e1);
```

$$qp := -\frac{q\,(e2 - e1)}{e2 + e1}$$

```
qpp := q*2*e2/(e2+e1);
```

$$qpp := 2\frac{q\,e2}{e2 + e1}$$

```
F1 := simplify((q + qp)/e1);
```

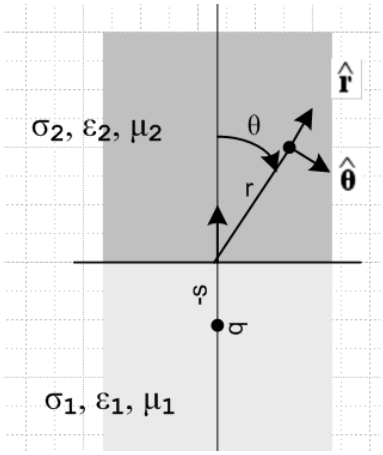
$$F1 := 2\frac{q}{e2 + e1}$$

```
F2 := qpp/e2;
```

$$F2 := 2\frac{q}{e2 + e1}$$

5. What did I do wrong in Attempt #1 and #2

In Attempt #1 I had this picture and I wrote



$$\varphi_1(r,\theta) = \Sigma_{n=0}^{\infty}[A_n r^n + B_n r^{-n-1}]\, P_n(\cos\theta)$$
$$\varphi_2(r,\theta) = \Sigma_{n=0}^{\infty}[C_n r^n + D_n r^{-n-1}]\, P_n(\cos\theta)$$

Then since nothing blows up at r = 0 in this picture, you get

$$\varphi_1(r,\theta) = \Sigma_{n=0}^{\infty}[A_n r^n]\, P_n(\cos\theta)$$
$$\varphi_2(r,\theta) = \Sigma_{n=0}^{\infty}[C_n r^n]\, P_n(\cos\theta)$$

But at infinity expect both φ = 0 and so you end up with

$$\varphi_1(r,\theta) = 0$$
$$\varphi_2(r,\theta) = 0$$

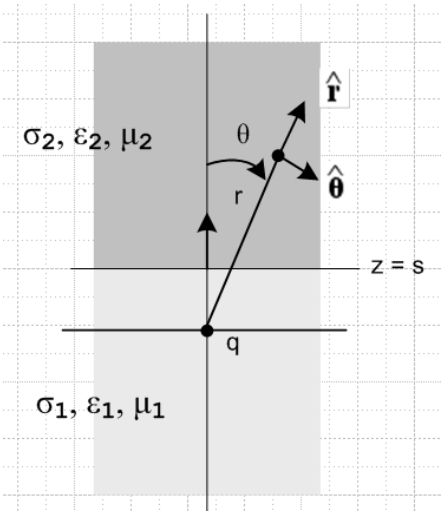
Here is what I did wrong, and I have made this error before in my recent electrostatics binge. You cannot represent the potential in region 1 in terms of the atoms shown because such atoms are for a region in which the Laplace equation is valid, and region 1 is not such a region since q is sitting there! But even ignoring that, you can see that nothing in the φ_1 expansion is going to give a blowup at the point charge

$$\varphi_1(s,\pi) = \Sigma_{n=0}^{\infty}[A_n s^n + B_n s^{-n-1}]\, P_n(-1) \qquad P_n(-1) = (-1)^n$$

Your only hope would be that this series diverges magically just at the point charge but not near it. But let's not waste more time on that.

What you COULD do and I may try below is write $\varphi_1 = q/R + \psi_1$ and then ψ_1 could have an atomic expansion. Then φ_1 can blow up as R→ 0.

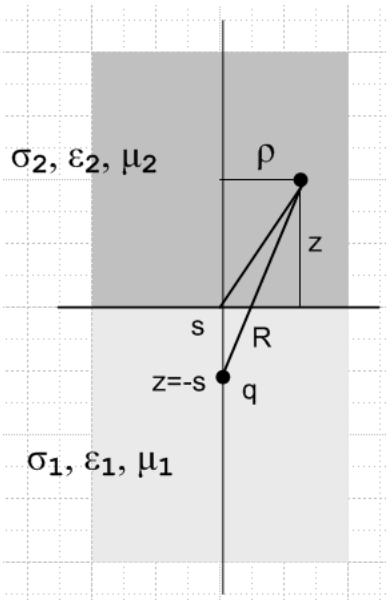
In Attempt #2 I just shifted the origin to get this picture,



but my error of using the atoms for region 2 is still present, so it is the same error really.

6. Attempt #1 in cylindrical coordinates

Sphericals are very clumsy for this problem because: if you put the point charge at the origin so it will have the simple q/r form, then the boundary surface does not have a constant coordinate. If you put the boundary at z = 0, then you need q/R where $\mathbf{R} = \mathbf{r} - s\,\hat{\mathbf{z}}$ which is messy. I have no idea why I used sphericals. The problem has azisym to cylindricals are the right coordinates to use. Then



$$R^2 = \rho^2 + (z+s)^2$$

Now you want to use cylindrical atoms and a point charge. Those atoms are:

$$e^{\pm kz}\,[\,J_m(k\rho),\, N_m(k\rho)]\;\; e^{\pm im\varphi}$$

where in our case we must have m = 0 and I guess k just floats as a real number for now

$$e^{\pm kz}\ [\ J_0(k\rho),\ N_0(k\rho)]$$

But the N_0 probably blows up at $\rho = 0$ which is along our innocent axis above, so we must have just

$$\varphi_1(\rho,z) = (1/4\pi\varepsilon_1)R^{-1} + \int_0^\infty dk\ A_k\ e^{kz}\ J_0(k\rho) \qquad \text{// always need z decay}$$

$$\varphi_2(\rho,z) = \int_0^\infty dk\ B_k\ e^{-kz}\ J_0(k\rho) \qquad R = \sqrt{\rho^2 + (z+s)^2}$$

This might be a bit clumsy due to the integrals, but I have done these things before.

Continuity of φ at the boundary gives

$$q/\sqrt{\rho^2 + s^2} + \int_0^\infty dk\ A_k\ J_0(k\rho) = \int_0^\infty dk\ B_k\ J_0(k\rho)$$

which is an integral condition on A_k and B_k.

Next, we need to compute the two field conditions at the boundary

$$\mathbf{E} = -\nabla\varphi = \hat{\boldsymbol{\rho}}\ \partial_\rho\varphi + \hat{\mathbf{z}}\ \partial_z\varphi = \hat{\boldsymbol{\rho}}\ E_\rho + \hat{\mathbf{z}}\ E_z$$

And then

$$E_{\rho 1} = \partial_\rho\varphi_1 = (1/4\pi\varepsilon_1)\ [\ -R^{-2}\ \partial_\rho R\ ] + \int_0^\infty dk\ A_k\ e^{kz}\ \partial_\rho\ J_0(k\rho)$$

But $\partial_\rho R = \rho/R$ and $\partial_\rho\ J_0(k\rho) = k\ J_0'(k\rho)$ so we have then

$$E_{\rho 1} = -\ (1/4\pi\varepsilon_1)\ \rho/R^3 + \int_0^\infty dk\ k\ A_k\ e^{kz}\ J_0'(k\rho)$$

Next,

$$E_{z1} = \partial_z\varphi_1 = (1/4\pi\varepsilon_1)\ [\ -R^{-2}\ \partial_z R\ ] + \int_0^\infty dk\ A_k\ \partial_z e^{kz}\ J_0(k\rho)$$

But $\partial_z R = 1/R * (z+s)$ so we get

$$E_{z1} = -\ (1/4\pi\varepsilon_1)(z+s)/R^3 + \int_0^\infty dk\ k\ A_k\ e^{kz}\ J_0(k\rho)$$

For region 2, we change A to B and omit the point charge, so here is our summary:

$$E_{\rho 1} = -\ (1/4\pi\varepsilon_1)\ \rho/R^3 + \int_0^\infty dk\ k\ A_k\ e^{kz}\ J_0'(k\rho)$$

$$E_{z1} = -\ (1/4\pi\varepsilon_1)(z+s)/R^3 + \int_0^\infty dk\ k\ A_k\ e^{kz}\ J_0(k\rho)$$

$$E_{\rho 2}(\rho,z) = \int_0^\infty dk\ k\ B_k\ e^{kz}\ J_0'(k\rho)$$

$$E_{z2}(\rho,z) = \int_0^\infty dk\ k\ B_k\ e^{kz}\ J_0(k\rho)$$

Now at z = 0 we have our boundary conditions. E_ρ are the tangential or parallels, E_z is the normal or perp. The rules are

$\varepsilon_2\ E_{n2} = \varepsilon_1\ E_{n1}$ n = normal to surface
$E_{t2} = E_{t2}$ t = tangential to surface

So therefore

$$\varepsilon_2\ E_{z2}(\rho,0) = \varepsilon_1\ E_{z1}(\rho,0)$$
$$E_{\rho 2}(\rho,0) = E_{\rho 1}(\rho,0)$$

which says

$$\varepsilon_2 \int_0^\infty dk\ k\ B_k\ J_0'(k\rho) = \varepsilon_1\ \{ -\ (1/4\pi\varepsilon_1)(s)/R^3 + \int_0^\infty dk\ k\ A_k\ J_0(k\rho)\ \}$$

$$\int_0^\infty dk\ k\ B_k\ J_0(k\rho) = -\ (1/4\pi\varepsilon_1)\ \rho/R^3 + \int_0^\infty dk\ k\ A_k\ J_0'(k\rho)$$

$$q/\sqrt{\rho^2 + s^2} + \int_0^\infty dk\ A_k\ J_0(k\rho) = \int_0^\infty dk\ B_k\ J_0(k\rho) \qquad R = \sqrt{\rho^2 + s^2}$$

On the last line we show the continuity condition.

Now pause to note that I could have used an expansion of the point charge of this form,

$$1/R = \Sigma_m\ e^{im\varphi}\ e^{-im\varphi'} \int_0^\infty dk\ \exp(-k|z-z'|)\ J_m(k\rho)\ J_m(k\rho')$$

If we put the point charge at z' = -s and ρ' = 0 and say φ' = 0 this says

$$1/R = \Sigma_m\ e^{im\varphi} \int_0^\infty dk\ \exp(-k|z+s|)\ J_m(k\rho)\ J_m(0)$$

or

10.7.3 $J_\nu(z) \sim (\frac{1}{2}z)^\nu / \Gamma(\nu+1), \quad \nu \neq -1, -2, -3, \ldots,$

This last says that at z = 0, $J_m(z) = 0$ for m > 0, and you have then only $J_0(0) = 1$, so

$$1/R = \Sigma_m e^{im\varphi} \int_0^\infty dk \exp(-k|z+s|) J_m(k\rho) \delta_{m,0} 1$$

or

$$1/R = \int_0^\infty dk \exp(-k|z+s|) J_0(k\rho)$$

I am not sure this would be very helpful due to the k+z combination.

So what do I do next? Maybe my choice of the $J_m(k\rho)$ was bad because this has to decay at large ρ.

10.7(ii) $z \to \infty$

When ν is fixed and $z \to \infty$,

10.7.8

$$J_\nu(z) = \sqrt{2/(\pi z)} \left(\cos(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) + e^{|\Im z|} o(1) \right),$$

at least it decays somewhat.

I think you have to apply $\int_0^\infty d\rho\ \rho\ J_0(k'\rho)$ to the conditions above in order to expose the coefficients, a sort of Hankel projection. For example,

$$\int_0^\infty dk\ k\ B_k\ J_0(k\rho) = -\ (1/4\pi\varepsilon_1)\ \rho/R^3 + \int_0^\infty dk\ k\ A_k\ J_0'(k\rho)$$

so

$$\int_0^\infty d\rho\ \rho J_0(k'\rho) \int_0^\infty dk\ k\ B_k\ J_0(k\rho)$$

$$= -\ (1/4\pi\varepsilon_1) \int_0^\infty d\rho\ \rho J_0(k'\rho)\ \rho/R^3 + \int_0^\infty d\rho\ \rho J_0(k'\rho) \int_0^\infty dk\ k\ A_k\ J_0'(k\rho)$$

Then you know that (transforms.doc)

$$\int_0^\infty d\rho\ \rho\ J_\nu(k\rho)\ J_\nu(k'\rho) = \delta(k-k')/k \qquad \text{// orthogonality}$$

and we use this with ν = 0 to get

$$\int_0^\infty d\rho\ \rho\ J_0(k\rho)\ J_0(k'\rho) = \delta(k-k')/k$$

Our condition above is then

$$\int_0^\infty dk\, k\, B_k \left[\int_0^\infty d\rho\, \rho J_0(k'\rho)\, J_0(k\rho) \right]$$

$$= - (1/4\pi\varepsilon_1) \int_0^\infty d\rho\, \rho J_0(k'\rho)\, \rho/R^3 + \int_0^\infty dk\, k\, A_k \int_0^\infty d\rho\, \rho J_0(k'\rho)\, J_0'(k\rho)$$

The LHS is then

$$\int_0^\infty dk\, k\, B_k \left[\delta(k-k')/k\right] = B_{k'}$$

and then we have

$$B_{k'} = - (1/4\pi\varepsilon_1) \int_0^\infty d\rho\, \rho J_0(k'\rho)\, \rho/R^3 + \int_0^\infty dk\, k\, A_k \int_0^\infty d\rho\, \rho J_0(k'\rho)\, J_0'(k\rho)$$

OK, enough on this huge distraction. It just goes to show how powerful the method of images really is, compared to this brute force approach. If you look at the expansions of the known solutions, you will find something like this

$$\varphi_2 = (1/4\pi\varepsilon_1) \left(q/R_1 + q'/R_2 \right) \qquad z > 0$$

$$\varphi_1 = (1/4\pi\varepsilon_2) \left(q''/R_1 \right) \qquad z < 0$$

where you would expand the 1/R objects as something like

$$1/R_1 = \int_0^\infty dk\, \exp(-k|z+s|)\, J_0(k\rho)$$

$$1/R_2 = \int_0^\infty dk\, \exp(-k|z-s|)\, J_0(k\rho)$$

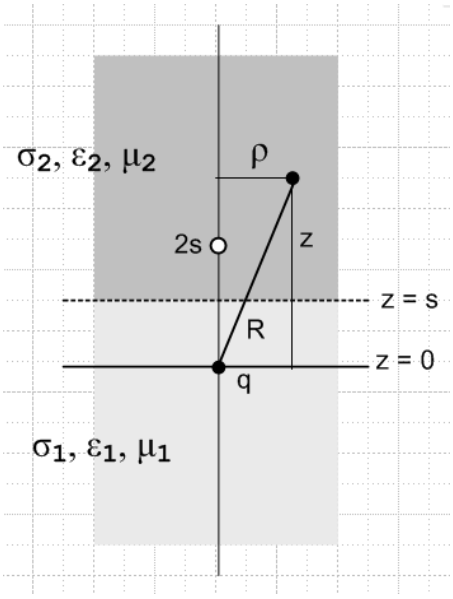
Let's go back to those original atoms,

$$e^{\pm kz} \left[J_0(k\rho), N_0(k\rho) \right]$$

Maybe I needed to divide space up into more strips, such as

7. Attempt #2 in cylindrical coordinates

Let's put the cylindrical origin at the point charge. Then



Atoms are again

$e^{\pm kz}$ [$J_0(k\rho)$, $N_0(k\rho)$]

Do our same expansions wrt this new origin,

$\varphi_1(\rho,z) = (1/4\pi\varepsilon_1)R^{-1} + \int_0^\infty dk\, A_k\, e^{kz}\, J_0(k\rho)$ // always need z decay

$\varphi_2(\rho,z) = \int_0^\infty dk\, B_k\, e^{-kz}\, J_0(k\rho)$ $R = \sqrt{\rho^2 + s^2}$

The expected solutions are

$\varphi_1 = (1/4\pi\varepsilon_1)\,(\ q/R_1 + q'/R_2)$ $z > s$

$\varphi^2 = (1/4\pi\varepsilon_2)\,(\ q''/R_1\ )$ $z < s$

and we know we can do this

$1/R = \int_0^\infty dk\, e^{-k|z|} J_0(k\rho)$

Using this, our expansions are

$\varphi_1(\rho,z) = (1/4\pi\varepsilon_1) \int_0^\infty dk\, e^{-k|z|} J_0(k\rho) + \int_0^\infty dk\, A_k\, e^{kz}\, J_0(k\rho)$

$$\varphi_2(\rho,z) = \int_0^\infty dk\, B_k\, e^{-kz}\, J_0(k\rho) \qquad\qquad R = \sqrt{\rho^2 + s^2}$$

and now things are all on the same footing! The **potential match** then says (set z = s > 0)

$$(1/4\pi\varepsilon_1) \int_0^\infty dk\, e^{-ks}\, J_0(k\rho) + \int_0^\infty dk\, A_k\, e^{ks}\, J_0(k\rho) = \int_0^\infty dk\, B_k\, e^{-ks}\, J_0(k\rho)$$

and maybe you can then argue that

$$(1/4\pi\varepsilon_1)\, e^{-ks} + A_k\, e^{ks} = B_k\, e^{-ks}$$

and there you have a nice clean relationship. I think this path will pan out. You have to do the field match conditions of course. Have I done this problem somewhere? Maybe, but let's finish it here.

Next, we need to compute the two field conditions at the boundary

$$\mathbf{E} = -\nabla\varphi = \hat{\boldsymbol{\rho}}\, \partial_\rho\varphi + \hat{\mathbf{z}}\, \partial_z\varphi = \hat{\boldsymbol{\rho}}\, E_\rho + \hat{\mathbf{z}}\, E_z$$

And we compute:

$$E_{\rho 1} = \partial_\rho\varphi_1 = (1/4\pi\varepsilon_1) \int_0^\infty dk\, k\, e^{-k|z|} J_0'(k\rho) + \int_0^\infty dk\, kA_k\, e^{kz}\, J_0'(k\rho)$$

$$E_{\rho 2} = \partial_\rho\varphi_2 = \int_0^\infty dk\, kB_k\, e^{-kz}\, J_0'(k\rho)$$

This is a tangential field as the picture shows, and our BC for it is that at z = s they are equal

$$(1/4\pi\varepsilon_1) \int_0^\infty dk\, k\, e^{-ks} J_0'(k\rho) + \int_0^\infty dk\, kA_k\, e^{ks}\, J_0'(k\rho) = \int_0^\infty dk\, kB_k\, e^{-ks}\, J_0'(k\rho)$$

and maybe we can conclude here that

$$(1/4\pi\varepsilon_1)k\, e^{-ks} + kA_k\, e^{ks} = kB_k\, e^{-ks}$$

or

$$(1/4\pi\varepsilon_1)e^{-ks} + A_k\, e^{ks} = B_k\, e^{-ks} \qquad \text{// same as matching potentials}$$

or

$$(1/4\pi\varepsilon_1)e^{-ks} = B_k\, e^{-ks} - A_k\, e^{ks}$$

which we see is the same as what we learned from equating the potentials. Hmmm... Next,

$$E_{z1} = \partial_z\varphi_1 = (1/4\pi\varepsilon_1) \int_0^\infty dk\, (\partial_z e^{-k|z|})^{z=s} J_0(k\rho) + \int_0^\infty dk\, A_k\, k\, e^{ks}\, J_0(k\rho)$$

$$E_{z2} = \partial_z\varphi_2 = \int_0^\infty dk\, B_k\, (-k)\, e^{-ks}\, J_0(k\rho)$$

This is a normal field, and our BC for it at z = s says

$$\varepsilon_1 \{ (1/4\pi\varepsilon_1) \int_0^\infty dk \, (\partial_z e^{-k|z|})^{z=s} J_0(k\rho) + \int_0^\infty dk \, A_k \, k \, e^{ks} J_0(k\rho)\}$$

$$= \varepsilon_2 \int_0^\infty dk \, B_k \, (-k) \, e^{-ks} J_0(k\rho)$$

and maybe we can conclude here that

$$\varepsilon_1 \{ (1/4\pi\varepsilon_1) (\partial_z e^{-k|z|})^{z=s} + A_k \, k \, e^{ks} \} = \varepsilon_2 B_k \, (-k) \, e^{-ks}$$

Now in the region of z = s, we have z > 0, so

$$(\partial_z e^{-k|z|})^{z=s} = (\partial_z e^{-kz})^{z=s} = -ke^{-ks}$$

so our last condition is then

$$\varepsilon_1 \{ (1/4\pi\varepsilon_1) (-ke^{-ks}) + A_k \, k \, e^{ks} \} = \varepsilon_2 B_k \, (-k) \, e^{-ks}$$

or

$$\varepsilon_1 \{ (1/4\pi\varepsilon_1) (ke^{-ks}) - A_k \, k \, e^{ks} \} = \varepsilon_2 B_k \, k \, e^{-ks}$$

or

$$\varepsilon_1 (1/4\pi\varepsilon_1) ke^{-ks} = \varepsilon_2 B_k \, k \, e^{-ks} + \varepsilon_1 A_k \, k \, e^{ks}$$

or

$$\varepsilon_1 (1/4\pi\varepsilon_1) e^{-ks} = \varepsilon_2 B_k \, e^{-ks} + \varepsilon_1 A_k \, e^{ks}$$

Thus, from our three boundary conditions, we end up with these two equations only:

$$(1/4\pi\varepsilon_1)e^{-ks} = B_k \, e^{-ks} - A_k \, e^{ks} \qquad (*)$$

$$\varepsilon_1 (1/4\pi\varepsilon_1) e^{-ks} = \varepsilon_2 B_k \, e^{-ks} + \varepsilon_1 A_k \, e^{ks}$$

This is 2 in 2, so we should have a nice solution coming in a moment. multiply top by ε_1

$$\varepsilon_1 (1/4\pi\varepsilon_1)e^{-ks} = \varepsilon_1 B_k \, e^{-ks} - \varepsilon_1 A_k \, e^{ks}$$

$$\varepsilon_1 (1/4\pi\varepsilon_1) e^{-ks} = \varepsilon_2 B_k \, e^{-ks} + \varepsilon_1 A_k \, e^{ks}$$

Add so the A_k terms cancel and we get

$$2 \, \varepsilon_1 (1/4\pi\varepsilon_1)e^{-ks} = (\varepsilon_1+\varepsilon_2) B_k \, e^{-ks}$$

$$B_k = (1/2\pi) / (\varepsilon_1+\varepsilon_2)$$

Then solve first equation (*) for A_k:

$$A_k e^{ks} = B_k e^{-ks} - (1/4\pi\varepsilon_1)e^{-ks}$$

$$A_k e^{ks} = (1/2\pi) / (\varepsilon_1+\varepsilon_2)e^{-ks} - (1/4\pi\varepsilon_1)e^{-ks}$$

$$4\pi\varepsilon_1 A_k e^{ks} = 2\varepsilon_1 / (\varepsilon_1+\varepsilon_2)e^{-ks} - e^{-ks}$$

$$(\varepsilon_1+\varepsilon_2)4\pi\varepsilon_1 A_k e^{ks} = 2\varepsilon_1 e^{-ks} - (\varepsilon_1+\varepsilon_2)e^{-ks}$$

$$(\varepsilon_1+\varepsilon_2)4\pi\varepsilon_1 A_k e^{ks} = (\varepsilon_1-\varepsilon_2)e^{-ks}$$

$$A_k = (1/4\pi\varepsilon_1)e^{-2ks} (\varepsilon_1-\varepsilon_2)/ (\varepsilon_1+\varepsilon_2)$$

Thus we think we have found our two unknown Smythian coefficients,

$A_k = (1/4\pi\varepsilon_1)e^{-2ks} [(\varepsilon_1-\varepsilon_2)/ (\varepsilon_1+\varepsilon_2)]$ // I got this right the first time.
$B_k = (1/4\pi\varepsilon_1) [2\varepsilon_1 / (\varepsilon_1+\varepsilon_2)]$ // first time had e^{-2ks} wrongly here

So let's install these into the potentials and see what we have:

$$\varphi_1(\rho,z) = (1/4\pi\varepsilon_1)R^{-1} + \int_0^\infty dk \{(1/4\pi\varepsilon_1)e^{-2ks} [(\varepsilon_1-\varepsilon_2)/ (\varepsilon_1+\varepsilon_2)]\} e^{kz} J_0(k\rho)$$

$$= (1/4\pi\varepsilon_1)R^{-1} + (1/4\pi\varepsilon_1) [(\varepsilon_1-\varepsilon_2)/ (\varepsilon_1+\varepsilon_2)] \int_0^\infty dk\, e^{-2ks+kz} J_0(k\rho)$$

The second term must be the image charge sitting in Region 2! Recall that

$$1/R|^{charge_at_a} = \int_0^\infty dk\, e^{-k|z-a|} J_0(k\rho)$$

Suppose this charge is at location z = 2s, just as a guess, then we have

$$1/R|^{charge_at_2s} = \int_0^\infty dk\, e^{-k|z-2s|} J_0(k\rho)$$

Observed from region 1, wherein z < s, we have

$z-2s < s-2s = -s$ $\Rightarrow z-2s < 0$ $\Rightarrow$ $|z-2s| = -(z-2s)$

Thus, viewed from Region 1 we have

$$1/R|^{charge_at_2s} = \int_0^\infty dk\, e^{+k(z-2s)} J_0(k\rho)$$

Thus we can write

$$\varphi_1(\rho,z) = (1/4\pi\varepsilon_1)R^{-1} + (1/4\pi\varepsilon_1)\,[(\varepsilon_1-\varepsilon_2)/(\varepsilon_1+\varepsilon_2)]\; 1/R|^{charge_at_2s}$$

and if we install charge q instead of q = 1 this says

$$\varphi_1(\rho,z) = (q/4\pi\varepsilon_1)R^{-1} + (q'/4\pi\varepsilon_1)\; 1/R|^{charge_at_2s}$$

$$q' = [(\varepsilon_1-\varepsilon_2)/(\varepsilon_1+\varepsilon_2)]q$$

and this agrees with Jackson's solution which I quote again

$$q' = -\left(\frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1}\right)q$$
$$q'' = \left(\frac{2\epsilon_2}{\epsilon_2 + \epsilon_1}\right)q \qquad (4.45)$$

Comment: The solution for φ_1 has ε_1 in both leading factors. This means all of space is filled with ε_1 and we have our image charge at z = 2s and the ε_2 space does not exist as far as the region 1 observer is concerned.

Now let's make sure we have the region 2 result correct as well.

$$A_k = (1/4\pi\varepsilon_1)e^{-2ks}\,[(\varepsilon_1-\varepsilon_2)/(\varepsilon_1+\varepsilon_2)]$$
$$B_k = (1/4\pi\varepsilon_1)\;[\;2\varepsilon_1/(\varepsilon_1+\varepsilon_2)]$$

$$\varphi_2(\rho,z) = \int_0^\infty dk\, B_k\, e^{-kz}\, J_0(k\rho)$$

$$= (1/4\pi\varepsilon_1)\,[\;2\varepsilon_1/(\varepsilon_1+\varepsilon_2)] \int_0^\infty dk\, e^{-kz}\, J_0(k\rho)$$

Now suppose we put a charge Q" at point z = 0 and view it from Region 2 where z > s. then

$$1/R|^{charge_at_a} = \int_0^\infty dk\, e^{-k|z-a|}J_0(k\rho)$$

$$1/R|^{charge_at_(0)} = \int_0^\infty dk\, e^{-k|z|}J_0(k\rho) = \int_0^\infty dk\, e^{-kz}J_0(k\rho)$$

Then we have shown that

$$\varphi_2(\rho,z) = (1/4\pi\varepsilon_1)\,[\;2\varepsilon_1/(\varepsilon_1+\varepsilon_2)]\; 1/R|^{charge_at_(0)}$$

If we call this charge Q" then we have

$\varphi_2(\rho,z) = Q''\ (1/4\pi\varepsilon_1)\ \ 1/R|^{charge_at_(0)}$

Again, the region 2 observer sees all space filled with ε_1 and sees this image at z = 0. The image is at the same position as the physical charge, but is Q" instead of q. All of our equations have $1/4\pi\varepsilon_1$ because that is how the original point charge entered the problem.

BUT, we can rewrite the above as follows:

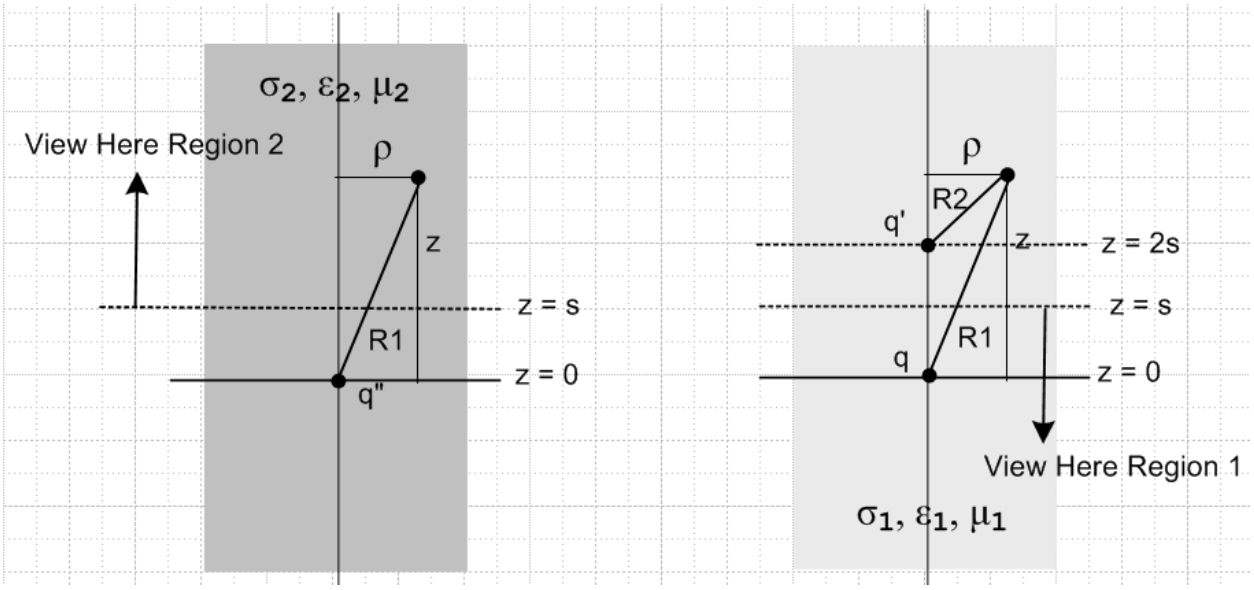
$$\varphi_2(\rho,z) = Q''\ (1/4\pi\varepsilon_1)\ \ 1/R_1 \ \ = [\ 2\varepsilon_1 / (\varepsilon_1+\varepsilon_2)]\ \ (1/4\pi\varepsilon_1)\ \ 1/R_1$$

$$= [\ 2\varepsilon_2 / (\varepsilon_1+\varepsilon_2)]\ \ (1/4\pi\varepsilon_2)\ \ 1/R_1$$

$$= q''\ (1/4\pi\varepsilon_2)\ \ 1/R_1$$

and this is the way Jackson presents it. The benefit of this approach is that we now have ε_2 matching the general ε in region 2. This seems much more reasonable.

Thus I have 100% confirmed Jackson's solution using cylindrical atoms in two regions with unknown Smythian coefficients. Here is what the solution looks like



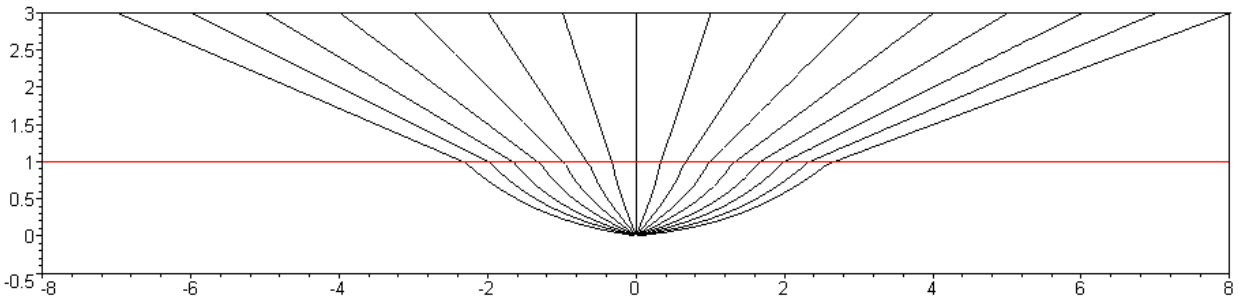
In Region 2 you just see a point charge, so E lines will be straight lines emanating from the origin,

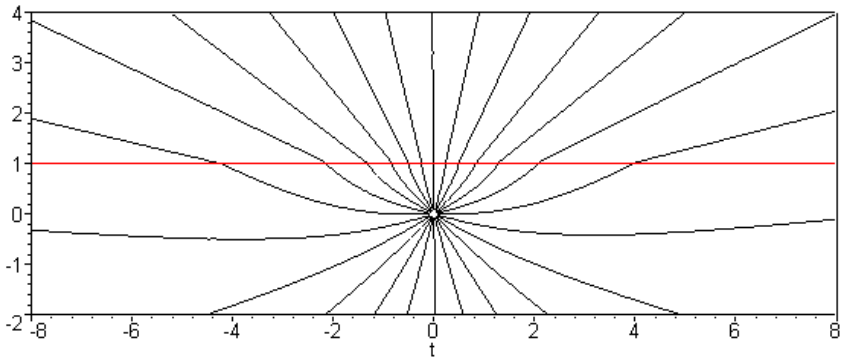
In Region 1 you get that same field plus the distorting dipole effect of a second charge and that makes the lines be curved in region 2.

8. Maple plots of the field lines

Here are plots I just made in Maple stealing my toroid code, for ε_1= 1 and ε_2 = 2:

File is charge in "2 diel.mws" and "2 diel v1.mws". In the first I start at z = 3 and work toward the point charge. In the second I start at points around a circle around the point charge and work out.





As the Bleaneys point out, the field lines are "refracted" at the boundary between the dielectrics.

9. The polarization charge density on the boundary, a Question, and a painful integral

Jackson computes this as

$$\sigma_{\text{pol}} = -\frac{q}{2\pi}\frac{\epsilon_0(\epsilon_2 - \epsilon_1)}{\epsilon_1(\epsilon_2 + \epsilon_1)}\frac{d}{(\rho^2 + d^2)^{3/2}} \qquad (4.47)$$

// units are OK

where his d is my s. I think if we include this $\sigma_{pol}(\xi)$ as a real charge, we could say that

$\varphi_1(\mathbf{r}) = E_1(\mathbf{0}|\mathbf{r}) + \int dS_\xi\, \sigma_{pol}(\xi)\, E_1(\xi|\mathbf{r})$ // in region 1

$\varphi_2(\mathbf{r}) = \int dS_\xi\, \sigma_{pol}(\xi)\, E_2(\xi|\mathbf{r})$

Question: In the region 1 solution picture, if we are in region 1, can we simply replace the image charge q' with a surface charge density σ_{pol} and everything stays the same in region 1 ? In other words, does the E field created by the polarization charge equal that created by the image charge q' as seen in region 1?

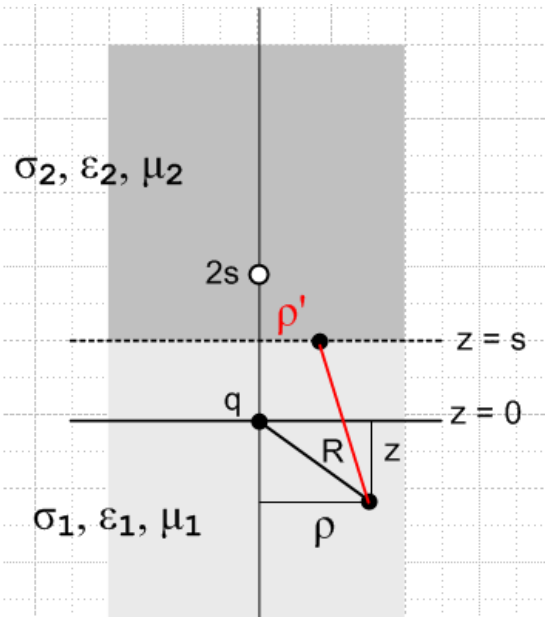
Answer: I think the answer is yes, but let's just calculate it.

$\sigma_{pol}(\rho) = \varepsilon_0(1/2\pi\varepsilon_1)\, q'\, s/\, (\rho^2+s^2)^{3/2}$ // units OK

$\varphi_{pol}(x) = \int dS_\xi\, E(x|\xi)\, \sigma_{pol}(\xi)$ $dS_\xi = \rho d\rho d\varphi$

$E(x|\xi) = (1/4\pi\varepsilon_1)/|\xi - x|$

Now this distance comes from this picture



Now in general the distance between two points in cylindrical coordinates is given by
[see study of 1/R for cylindricals]

$D^2 = \rho^2 + \rho'^2 + (z-z')^2 - 2\rho\rho'\cos(\varphi-\varphi')$

In our picture above we take the upper red line endpoint as the primed location, so

$D^2 = \rho^2 + \rho'^2 + (z-s)^2 - 2\rho\rho'\cos(\varphi-\varphi')$

$D = |\xi - x| = \sqrt{\rho^2 + \rho'^2 + (z-s)^2 - 2\rho\rho'\cos(\varphi-\varphi')}$

Then we have this result [I later learned that σ_{pol} propagates with $1/4\pi\varepsilon_0 R$!]

$$\sigma_{\text{pol}} = -\frac{q}{2\pi}\frac{\epsilon_0(\epsilon_2 - \epsilon_1)}{\epsilon_1(\epsilon_2 + \epsilon_1)}\frac{d}{(\rho^2 + d^2)^{3/2}} \qquad (4.47)$$

$$\varphi_{pol}(x) = \int dS_\xi \, E(x|\xi)\, \sigma_{pol}(\xi)$$

$$= \int \rho d\rho \int d\varphi \; [(1/4\pi\varepsilon_0)\, 1/D\,]\, \sigma_{pol}(\rho)$$

$$= \int \rho' d\rho' \int d\varphi \; [(1/4\pi\varepsilon_0)\, 1/D\,]\, \varepsilon_0 (1/2\pi\varepsilon_1)\, q'\, s/\, (\rho'^2+s^2)^{3/2}$$

$$= (1/2\pi\varepsilon_1)\,(1/4\pi)\, q'\, s \int \rho' d\rho' \int d\varphi \; [1/\sqrt{\rho^2 + \rho'^2 + (z-s)^2 - 2\rho\rho'\cos(\varphi-\varphi')}\;]\, /\, (\rho'^2+s^2)^{3/2}$$

Now this potential is symmetric in both directions, ± z, in its contribution to φ, because as just noted, the propagator either way is $(1/4\pi\varepsilon_0)$.

Now this is how things must work out. If we are in region 1, the potential φ_{pol} must be the same as that of an image charge q' at z = 2s immersed in an all ε_1 world, and we would get

$$\varphi_{pol}(x) = q'\,(1/4\pi\varepsilon_1)\; 1/R|^{z=2s} = q'\,(1/4\pi\varepsilon_1) 1/\sqrt{(2s-z)^2 + \rho^2} \qquad \text{Region 1}$$

On the other hand, if we are in region 2, the potential φ_{pol} must be the same as that of an image charge q-q" at the origin immersed in an all ε_2 world, and we would get

$$\varphi_{pol}(x) = (q-q")\,(1/4\pi\varepsilon_2)\; 1/R|^{z=0} = (q-q")\,(1/4\pi\varepsilon_2)\; 1/\sqrt{z^2 + \rho^2} \qquad \text{Region 2}$$

Somehow the above integral produces both these results, where the difference is in sign(z-s). I have seen how this fancy stuff arises before.

To make this fly for Region 1, the following would have to be true

$$q'\,(1/4\pi\varepsilon_1) 1/\sqrt{(2s-z)^2 + \rho^2}$$

$$= (1/4\pi\varepsilon_1)\,(1/2\pi)\, q'\, s \int \rho' d\rho' \int d\varphi \; [1/\sqrt{\rho^2 + \rho'^2 + (z-s)^2 - 2\rho\rho'\cos(\varphi-\varphi')}\;]\, /\, (\rho'^2+s^2)^{3/2}$$

or

$$1/\sqrt{(2s-z)^2 + \rho^2}$$

$$= (1/2\pi)\, s \int \rho' d\rho' \int d\varphi \; [1/\sqrt{\rho^2 + \rho'^2 + (z-s)^2 - 2\rho\rho'\cos(\varphi-\varphi')}\;]\, /\, (\rho'^2+s^2)^{3/2}$$

$$= (s/2\pi) \int \rho' d\rho' \, [1/(\rho'^2+s^2)^{3/2}] \int d\varphi \; [1/\sqrt{\rho^2 + \rho'^2 + (z-s)^2 - 2\rho\rho'\cos(\varphi-\varphi')}\;]$$

These are famous integrals I have messed with many times.

$$2.^* \qquad \int_0^{\pi} \frac{dx}{\sqrt{a \pm b\cos x}} = \int_{-\pi/2}^{\pi/2} \frac{dx}{\sqrt{a \pm b\sin x}} = \frac{2}{\sqrt{a+b}}\, \boldsymbol{E}\left(\sqrt{\frac{2b}{a+b}}\right)$$

$$[a > b > 0]$$

(2) Page 408 (major). I think that the bold E and bold K are swapped between the two integrals in section 3.670. The integral 3.670.1 should have bold E, and the integral 3.670.2 should have bold K.

If I stumble upon more errata, I will send you a longer cumulative list.

Best regards and thanks for doing this very useful task of getting the bugs out of this valuable resource,

and this is right where I found the bug in GR7 (never got a response). So assume it is a K. Then

$a = \rho^2+\rho'^2+(z-s)^2$ $\quad b = 2\rho\rho' > 0$ $\quad$ assume - sign in quote above

$a + b = (\rho+\rho')^2 + (z-s)^2$

Then I claim

$$\int d\varphi'[1/\sqrt{\rho^2 + \rho'^2 + (z-s)^2 - 2\rho\rho'\cos(\varphi-\varphi')}\ ]$$

$$= 2*2\ 1/\sqrt{(\rho+\rho')^2 + (z-s)^2}\ K\left(\sqrt{\frac{4\rho\rho'}{(\rho+\rho')^2 + (z-s)^2}}\ \right)$$

The ρ' integral is then very painful, but I know I have done all this before. We have to show that

$$2\pi/\sqrt{(2s-z)^2 + \rho^2} = s \int \rho' d\rho' \left[\ 1/\ (\rho^2+s^2)^{3/2}\right] 4\ [1/\sqrt{(\rho+\rho')^2 + (z-s)^2}\ ]\ K\left(\sqrt{\frac{4\rho\rho'}{(\rho+\rho')^2 + (z-s)^2}}\ \right)$$

Are the dimensions right? LHS = 1/L RHS = L L^2 $1/L^3$ 1/L = 1/L so yes. Maybe replace y = z-s

$$2\pi/\sqrt{(y-s)^2 + \rho^2} = 4s \int \rho' d\rho' \left[\ 1/\ (\rho'^2+s^2)^{3/2}\right] [1/\sqrt{(\rho+\rho')^2 + y^2}\ ]\ K\left(\sqrt{\frac{4\rho\rho'}{(\rho+\rho')^2 + y^2}}\ \right)$$

when $y = z-s < 0$, and the left side is different when $y > 0$.

It will turn out that both sides are functions of a complex variable y which has cuts, etc.

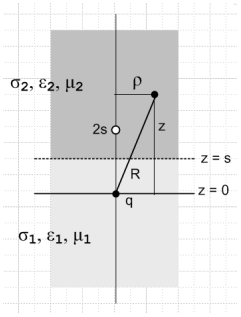
This would take me a very long time to get figured out, so let's stop. I think the claim is correct that the polarization charge does create a field which looks like the point charge q' field.

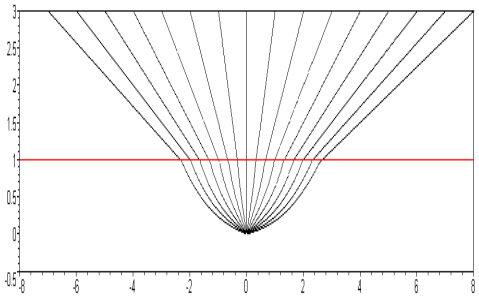
10. A propagator interpretation of the above results

Each region has its own Poisson equation

$\nabla^2 u = \rho/\varepsilon_1$ region 1 $z < s$

$\nabla^2 u = 0$ region 2 $z > s$





Reg 1 bottom

Consider the following canonical equation form

$$u_1(x) = \int_R d\xi\, g_1(x|\xi)\, q(\xi) - \int_\sigma dS_\xi\, f(\xi)\, \partial_{\xi n} g_1(x|\xi) .$$

If we regard the z = 0 plane as a boundary, then $g_1(x|\xi)$ would have to vanish on that boundary and I know how to do that by putting an exact opposite mirror charge at z = 2s. That is to say

$$g_1(x|\xi) = (1/4\pi\varepsilon_1)\ 1/R|^{z=0} - (1/4\pi\varepsilon_1)\ 1/R|^{z=2s}$$

This of course also vanishes on the lower great half sphere. And $\nabla^2 g_1(x|\xi) = -\delta(x-0)/\varepsilon_1$ because the second term has no hit in region 1. Then $\partial_{\xi n} g_1(x|\xi)$ is the normal E field at z = s which would be twice the normal component of just the lower point charge on that plane.

In region 2 there exists no charge ρ, so we get just

$$u_2(x) = - \int_\sigma dS_\xi\, f(\xi)\, \partial_{\xi n} g_2(x|\xi) .$$

I think the correct g_2 for this case would be

$$g_2(x|\xi) = (1/4\pi\varepsilon_2)\ 1/R|^{z=2s} - (1/4\pi\varepsilon_2)\ 1/R|^{z=0}$$

which is the solution IF we put a unit charge at z = 2s in region 2.

But I don't know the value of f(ξ) on the boundary. It is not a metallic boundary.

Well of course I really do know φ on the boundary because I know the solution of the problem. For example, in region 2 I know that

$$\varphi_2(\rho,z) = q''\ (1/4\pi\varepsilon_2)\ \ 1/R|^{charge_at_(0)} = q''\ (1/4\pi\varepsilon_2)\ \ 1/R_1 \qquad R_1 = \sqrt{\rho^2+z^2}$$

Then

$$\varphi_2(\rho,s) = q'' \, (1/4\pi\varepsilon_2) \; 1/\sqrt{\rho^2+s^2} \quad = f(\xi)$$

which varies with ρ and is thus not a constant.

In this last discussion, the "propagator" g_1 or g_2 is a boundary-specific propagator, not the free-ε propagator.

11. Conjecture

I think in each region the potential is created entirely by the physical point charge and the physical polarization charge on the surface, and in that context you can ignore everything else and use the Coulomb law integral with the appropriate ε for the region. Of course that integral may be a complete nightmare to compute.