Electrostatics: Potential inside a ring of charge PhL 11.12.13

I think my treatments below are viable. It is not clear to me now 2/9/14 *why* I was interested in this problem at the time. I guess if you think of the mag surface current K_z on a piece of round conductor, it is a like a ring of charge??? I have totally lost the connection and the edit log for this date is also confusing. But I think the work here regarding this Jackson problem is probably all OK. I might file a copy of this in electrostatics somewhere.

My interest in this problem arose recently when doing the transmission line Chapter 4 work where I mistakenly used the 3D propagator instead of the 2D one. I was reminded of my GR7 errata with the two integrals, and it all has a bearing on this simple problem.

We want to look at this 3D problem solved a two different ways. Using spherical atoms, Jackson blue provides a solution at the top of page 104. I see he has dropped equation numbers on equations that perhaps are just applications. I will quote this result in a while. [This is also in green in cgs.]

In Part 1 I do it my way and get a K function
In Part 2 I do it Jackson's way and get a spherical coordinate solution.
// I never show that the two solutions in general are equivalent.
// But I do show that they agree at the center of the ring
In Part 3 I plot my solution, it is a very simple plot in the plane only, log diverge at the ring
In Part 4 I provide "atomic support" for the Jackson method
In Part 5 I do the 2D ring charge problem which is a completely different animal.

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1. The Poisson Integral Solution to the 3D ring charge problem.

My interest however is the Poisson Integral solution. Start with

$$\nabla^2\varphi = -\rho/\varepsilon_0 \qquad \rho = \rho_0\delta(r-a)\delta(z) = \text{ring of charge}$$

Then I expect to have, using cylindrical coordinates,

$$\varphi(\mathbf{x}) = (1/4\pi)\,(1/\varepsilon_0)\int dV'\,\rho(\mathbf{x}')/R = (1/4\pi)\,(\rho_0/\varepsilon_0)\int_0^\infty r'dr'\int d\theta'\int_0^\infty dz'\,\delta(r'-a)\delta(z')/R$$

$$= (a/4\pi)\,(\rho_0/\varepsilon_0)\int d\theta'\,/R$$

where

$$R^2 = [r^2 + a^2 + z^2] - 2a\cos(\theta-\theta') = A^2 - 2a\cos(x)$$

Our result is then

$$\varphi(x) = (a/4\pi)(\rho_0/\varepsilon_0) \int_{-\pi}^{\pi} dx \; 1/\sqrt{A^2 - 2a\cos x} = 2(a/4\pi)(\rho_0/\varepsilon_0) \int_{0}^{\pi} dx \; 1/\sqrt{A^2 - 2a\cos x}$$

This integral appears wrong TWICE in GR7, I did not notice that until today. Here are the correct integrals from my note:

$$\int_0^{\pi} dx(1/\sqrt{a \pm b\cos x}) = (2/\sqrt{a+b})\, K(\sqrt{2b/(a+b)}) \qquad a > b > 0$$

$$\int_0^{\pi} dx \sqrt{a \pm b\cos x} = (2\sqrt{a+b})\, E(\sqrt{2b/(a+b)}) \qquad a > b > 0$$

Now in my application of this second integral, I have $a = A^2 = r^2 + a^2 + z^2$ and $b = 2ar$. This means

$$a+b = r^2 + a^2 + z^2 + 2ar = (r+a)^2 + z^2$$

The solution is then $\sqrt{(r+a)^2 + z^2}$

$$\varphi(\mathbf{x} = r,\theta,z) = 2(a/4\pi)(\rho_0/\varepsilon_0) \frac{2}{\sqrt{(r+a)^2 + z^2}} K\left(\sqrt{\frac{4ar}{(r+a)^2 + z^2}}\right)$$

or

$$\varphi(\mathbf{x} = r,\theta,z) = (a/\pi)(\rho_0/\varepsilon_0) \frac{1}{\sqrt{(r+a)^2 + z^2}} K\left(\sqrt{\frac{4ar}{(r+a)^2 + z^2}}\right)$$

Dimensions of ρ_0 are C/m (linear charge density) while ε_0 = farad/m so

$$\dim(\varphi) = m * C/m * m/\text{farad} * m^{-1} = C/F = \text{volt}$$

I am now going to change the symbols in my cylindrical coordinates solution to be

$$\varphi(\mathbf{x} = \rho,\varphi,z) = (a/\pi)(\rho_0/\varepsilon_0) \frac{1}{\sqrt{(\rho+a)^2 + z^2}} K\left(\sqrt{\frac{4a\rho}{(\rho+a)^2 + z^2}}\right)$$

Now I have $2\pi a\rho_0 = q$, the total charge of my ring, so

$$(a/\pi)(\rho_0/\varepsilon_0) = (a/\pi)(q/2\pi a\varepsilon_0) = q/(2\pi^2\varepsilon_0)$$

and then my solution is

$$\varphi(\mathbf{x} = \rho,\varphi,z) = q/(2\pi^2\varepsilon_0)\ \frac{1}{\sqrt{(\rho+a)^2 + z^2}}\ K\left(\sqrt{\frac{4a\rho}{(\rho+a)^2 + z^2}}\right) \qquad \text{// ring at } z = 0$$

As expected, things are independent of φ. Now I want to make one more change: I will put my ring of charge at z = b to match Jackson below. Then

$$\varphi(\mathbf{x} = \rho,\varphi,z) = q/(2\pi^2\varepsilon_0)\ \frac{1}{\sqrt{(\rho+a)^2 + (z-b)^2}}\ K\left(\sqrt{\frac{4a\rho}{(\rho+a)^2 + (z-b)^2}}\right)$$

2. The Jackson Solution to the 3D ring charge problem

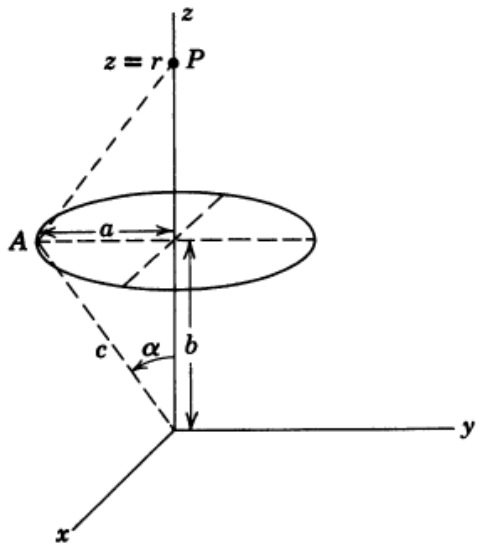


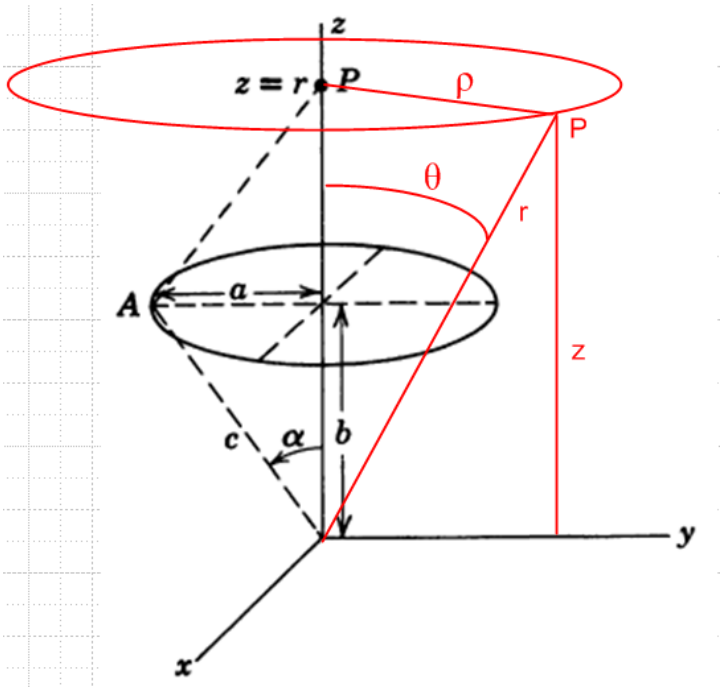
Figure 3.4 Ring of charge of radius a and total charge q located on the z axis with center at $z = b$.

$$\Phi(r,\ \theta) = \frac{q}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\alpha) P_l(\cos\theta)$$

where $r_<$ ($r_>$) is the smaller (larger) of r and c.

For some reason Jackson writes his z value as r. His solution is actually quite unclear because he doesn't really say what "r" is for his *general* solution I quoted above. I presume it is the distance of the observation point from his picture's origin. I am guessing that r,θ are two *spherical* coordinates since I see spherical atoms in his formula, and then there is no φ dependence. My solution is more geared to cylindrical coordinates.

Now do we relate Jackson's r and θ to my ρ and z ?



$r^2 = \rho^2 + z^2$

$z = r\cos\theta = \sqrt{\rho^2+z^2}\cos\theta \quad \Rightarrow \quad \cos\theta = z/\sqrt{\rho^2+z^2}$

Then Jackson also has $\cos\alpha = b/c = b/\sqrt{a^2+b^2}$ which is just a constant. Then

$r_< = \min(r, \sqrt{a^2+b^2}) = \min(\sqrt{\rho^2+z^2}, \sqrt{a^2+b^2})$
$r_> = \max(r, \sqrt{a^2+b^2}) = \max(\sqrt{\rho^2+z^2}, \sqrt{a^2+b^2})$

Then I can write his solution as

$$\varphi_J = [q/4\pi\varepsilon_0] \Sigma_{n=0}^{\infty} [\min(\sqrt{\rho^2+z^2}, \sqrt{a^2+b^2})]^n [\max(\sqrt{\rho^2+z^2}, \sqrt{a^2+b^2})]^{-n-1}$$

$$P_n(b/\sqrt{a^2+b^2}) P_n(z/\sqrt{\rho^2+z^2})$$

which I can compare to my solution

$$\varphi_L(\mathbf{x} = \rho,\varphi,z) = q/(2\pi^2\varepsilon_0) \frac{1}{\sqrt{(\rho+a)^2 + (z-b)^2}} K\left(\sqrt{\frac{4a\rho}{(\rho+a)^2 + (z-b)^2}}\right)$$

<u>Let's examine both solutions in the plane of the ring</u>. For me, z = b. For him, we still have to do the min max depending no whether we are inside or outside the ring. So

$$\varphi_J = [q/4\pi\varepsilon_0] \Sigma_{n=0}^{\infty} [\min(\sqrt{\rho^2+b^2}, \sqrt{a^2+b^2})]^n [\max(\sqrt{\rho^2+b^2}, \sqrt{a^2+b^2})]^{-n-1}$$

$$P_n(b/\sqrt{a^2+b^2}\,)\,P_n(b/\sqrt{\rho^2+b^2}\,)$$

$$\varphi_L = q/(2\pi^2\varepsilon_0)\,\frac{1}{\rho+a}\,K\left(\frac{\sqrt{4a\rho}}{\rho+a}\right)$$

Now at ρ = a, I have K(2a/2a) = K(1) which is no doubt a singular point for K. Now let's restrict interest to "inside the ring". Then $\sqrt{\rho^2+b^2} < \sqrt{a^2+b^2}$ and we get

$$\varphi_J = [q/4\pi\varepsilon_0]\,\Sigma_{n=0}^{\infty}\,(\sqrt{\rho^2+b^2}\,)^n\,(\sqrt{a^2+b^2}\,)^{-n-1}\;P_n(b/\sqrt{a^2+b^2}\,)\,P_n(b/\sqrt{\rho^2+b^2}\,)$$

$$\varphi_L = q/(2\pi^2\varepsilon_0)\,\frac{1}{\rho+a}\,K\left(\frac{\sqrt{4a\rho}}{\rho+a}\right)$$

Now how about right at the center of the ring:

$$\varphi_J = [q/4\pi\varepsilon_0]\,\Sigma_{n=0}^{\infty}\,b^n\,(\sqrt{a^2+b^2}\,)^{-n-1}\;P_n(b/\sqrt{a^2+b^2}\,)\,P_n(1)$$

$$\varphi_L = q/(2\pi^2\varepsilon_0)\,\frac{1}{a}\,K(0)$$

Now look up K(0) in AS special values (I hope)

19.6(i) Complete Elliptic Integrals

19.6.1 $$K(0) = E(0) = K'(1) = E'(1) = \tfrac{1}{2}\pi,$$

So then my solution becomes

$$\varphi_L = q/(2\pi^2\varepsilon_0)\,\frac{1}{a}\,(\pi/2) = [\,1/4\pi\varepsilon_0\,]\,q/a$$

which is of course the correct result since all parts of the ring are distance a away.

Jackson meanwhile gets

$$\varphi_J = [q/4\pi\varepsilon_0]\,\Sigma_{n=0}^{\infty}\,b^n\,(\sqrt{a^2+b^2}\,)^{-n-1}\;P_n(b/\sqrt{a^2+b^2}\,)$$

which *seems* to depend on b, but as I will show below, it does not in fact depend on b. First, for b = 0 it gives the right answer since then only the n=0 term survives and

$$\varphi_J = [q/4\pi\varepsilon_0]\,1/a\,P_0(0) = [q/4\pi\varepsilon_0]\,1/a$$

In order for this result to be correct for any b, one would have to show that

$\Sigma_{n=0}^{\infty} b^n \left(\sqrt{a^2+b^2}\right)^{-n-1} P_n\left(b/\sqrt{a^2+b^2}\right) = 1/a$ for any b !! (*)

Maple seems to indicate that this is true! Rewrite differently,

$x \equiv b/\sqrt{a^2+b^2} < 1 \quad \Rightarrow \quad \sqrt{a^2+b^2} = b/x \quad a^2 + b^2 = b^2/x^2$

$\Rightarrow \quad a^2 = b^2(1/x^2-1) = b^2(1-x^2)/x^2$

$\Rightarrow \quad b^2 = a^2x^2/(1-x^2)$

$\Rightarrow \quad (b/a)^2 = x^2/(1-x^2)$

$\Rightarrow \quad (b/a) = x/\sqrt{1-x^2}$

$\Sigma_{n=0}^{\infty} b^n (b/x)^{-n-1} P_n(x) = 1/a$?

$\Sigma_{n=0}^{\infty} b^n b^{-n-1} x^{n+1} P_n(x) = 1/a$?

$\Sigma_{n=0}^{\infty} b^{-1} x^{n+1} P_n(x) = 1/a$?

$b^{-1}\Sigma_{n=0}^{\infty} x^{n+1} P_n(x) = 1/a$?

$\Sigma_{n=0}^{\infty} x^{n+1} P_n(x) = b/a = x/\sqrt{1-x^2}$?

$\Sigma_{n=0}^{\infty} x^n P_n(x) = 1/\sqrt{1-x^2}$?

Now there is the following generating function sum rule:

8.92 Series of Legendre polynomials

8.921 The generating function:

$$\frac{1}{\sqrt{1-2tz+t^2}} = \sum_{k=0}^{\infty} t^k P_k(z) \qquad \left[|t| < \min\left|z \pm \sqrt{z^2-1}\right|\right]$$

$$= \sum_{k=0}^{\infty} \frac{1}{t^{k+1}} P_k(z) \qquad \left[|t| > \max\left|z \pm \sqrt{z^2-1}\right|\right]$$

If I set t = z = x this says

$1/\sqrt{1-x^2} = \Sigma_{n=0}^{\infty} x^n P_n(x) \qquad |x| < \min\left| x \pm \sqrt{x^2-1} \right| = \min\left| x \pm i\sqrt{1-x^2} \right| \qquad (**)$

But

$\left| x \pm i\sqrt{1-x^2} \right|^2 = (x \pm i\sqrt{1-x^2})(x \mp i\sqrt{1-x^2}) = x^2 - i^2(1-x^2) = x^2+(1-x^2) = 1$

Thus I have shown that

$$|\, x \pm i\sqrt{1-x^2}\,| = 1 \qquad \Rightarrow \qquad \min |\, x \pm i\sqrt{1-x^2}\,| = 1 \qquad \Rightarrow \qquad |x| < 1$$

and therefore the sum (**) is valid in my application. Therefore the last ? line above is true, and therefore (*) is true in fact for any real b.

3. More on My Solution to the 3D ring charge problem

If the ring is at z = 0, my solution is this:

$$\varphi(\mathbf{x} = \rho,\varphi,z) = q/(2\pi^2\varepsilon_0)\ \frac{1}{\sqrt{(\rho+a)^2 + z^2}}\ K\left(\sqrt{\frac{4a\rho}{(\rho+a)^2 + z^2}}\right) \qquad \text{// ring lies at } z = 0$$

If we are interested in φ inside the ring itself, we set z = 0 to get

$$\varphi = q/(2\pi^2\varepsilon_0)\ \frac{1}{\rho+a}\ K\left(\frac{\sqrt{4a\rho}}{\rho+a}\right) \qquad 0 < \rho < a$$

where the argument of K runs 0 to 1. Rewrite this as

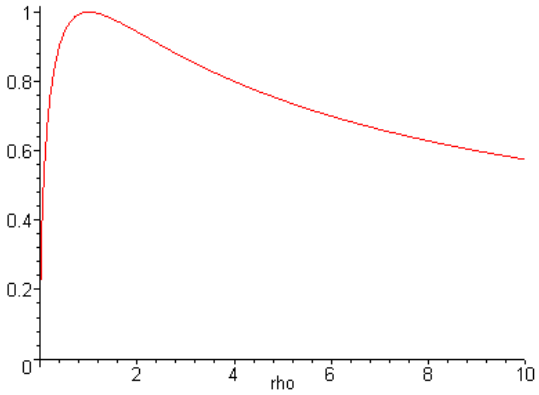
$$\varphi = q/(4\pi\varepsilon_0)\ \left[\ (2/\pi)\ \frac{1}{\rho+a}\ K\left(\frac{\sqrt{4a\rho}}{\rho+a}\right)\ \right]$$

If we plot the argument of K versus ρ we find

```
restart;
z := sqrt(4*a*rho)/(a+rho);
```

$$z := 2\,\frac{\sqrt{a\,\rho}}{a+\rho}$$

```
a := 1:
plot(z,rho = 0..10);
```



which tells us that as ρ ranges from ρ= 0 to ρ = ∞, the K argument is always ≤ 1. It is only = 1 at the point where ρ = a. So we don't need any fancy K transformation formulas.

For small ρ we have already shown that [...] → 1/a as expected.

For large ρ we have

$$K\left(\frac{\sqrt{4a\rho}}{\rho+a}\right) \approx K\left(\frac{\sqrt{4a\rho}}{\rho}\right) = K\left(\sqrt{4a/\rho}\right) \approx K(0) = \pi/2$$

and then

$$\left[(2/\pi)\ \frac{1}{\rho+a} K\left(\frac{\sqrt{4a\rho}}{\rho+a}\right) \right] \approx (2/\pi)\ (1/\rho)\ (\pi/2) = 1/\rho$$

and then

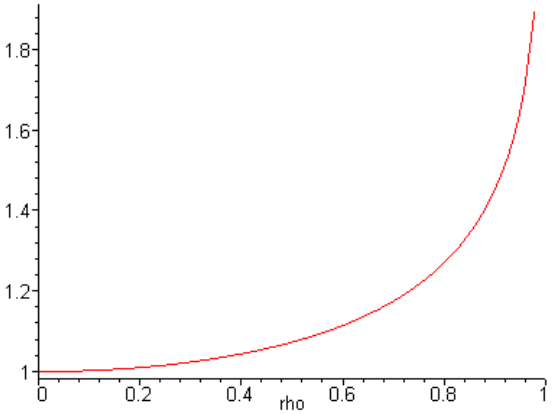
$\varphi = q/(4\pi\varepsilon_0)\ (1/\rho)$ // as expected.

Here is a Maple plot for ρ in (0,a):

```
phi :=(2/Pi)*(1/(rho+a))*EllipticK(sqrt(4*a*rho)/(rho+a));
```

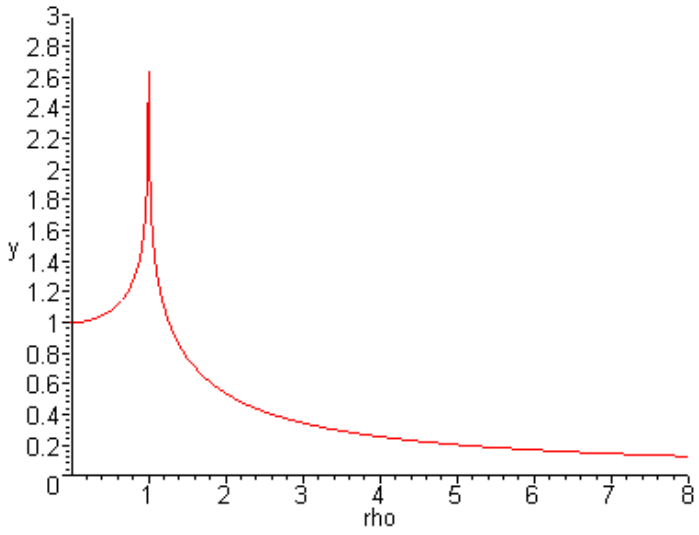
$$\phi := 2\,\frac{\text{EllipticK}\left(2\,\frac{\sqrt{a\,\rho}}{\rho+a}\right)}{\pi\,(\rho+a)}$$

```
a := 1:
plot(phi,rho = 0..1);
```



At ρ = 0 we know the result is [...] = 1/a which in our plot is just 1.
At ρ→a we expect logarithmic divergence since it is like a point next to an infinite line charge which has the general behavior (1/2π) ln(1/(a-ρ)) where a-ρ = r, and the 2D propagator in Stak is (1/2π)ln(1/r). I can in fact plot right through the singular point like so

```
plot(phi,rho = 0..8, y = 0..3);
```



and then you see the usual 1/r drop off.

Conclusion: I think my expression for φ of the ring change in terms of the K function is correct.

4. Jackson's method written out

For this azisym problem, we know what the atoms are, so

$$\varphi(r,\theta) = \Sigma_{n=0}^{\infty} A_n r^{-n-1} P_n(\cos\theta) \qquad \text{large r}$$

He computes things for θ = 0 and finds there that

$$\varphi(r,\theta=0) = \Sigma_{n=0}^{\infty} [(q/4\pi\varepsilon_0)c^n P_n(\cos\alpha)] r^{-n-1} P_n(1) \qquad \text{large r}$$

where $P_n(1) = 1$. This then determines that

$$A_n = (q/4\pi\varepsilon_0)c^n P_n(\cos\alpha)$$

and then the full result must be

$$\varphi(r,\theta) = \Sigma_{n=0}^{\infty} (q/4\pi\varepsilon_0)c^n P_n(\cos\alpha) r^{-n-1} P_n(\cos\theta) \qquad \text{large r}$$

He does a similar thing for small r and there you have it. I think his presentation is a little strange.

5. The 2D ring charge problem

I don't recall ever doing this problem before. I scanned Stak Chapter 6 and did not notice it. The result I get here seems reasonable in various ways. So here is the derivation:

Consider the general 2D situation

$$\nabla^2_{2D}\ \varphi = -\rho/\varepsilon_0$$

The potential of a unit positive point charge is $\varphi = (1/4\pi\varepsilon_0)(1/R)$ in 3D. For Stak, the ε_0 is just missing in the bottom, as on his page 50 (5.103). So I would say that in 2D the potential of a unit point charge at the origin is given by

$$\varphi = (1/2\pi\varepsilon_0)\ln(1/r). \qquad (*)$$

This diverges logarithmically at both r = 0 and r = ∞. I think Stak gets this by assuming the divergence theorem div **E** = $-\rho/\varepsilon_0$ in two dimensions and maintaining **E** = $-\nabla\varphi$.

This is also the Green's Function, so our solution is then

$$\varphi(\mathbf{x}) = (1/2\pi\varepsilon_0)\int dA'\ \rho(\mathbf{x}')\ \ln(1/R) \qquad R = |\mathbf{x} - \mathbf{x}'| \qquad R^2 = (x-x')^2 + (y-y')^2$$

Writing this in polar coordinates

$$\varphi(\mathbf{x}) = (1/2\pi\varepsilon_0)\int r'dr' \int d\theta'\ \rho(r',\theta')\ \ln(1/R) \qquad R^2 = r^2+r'^2 - 2rr'\cos(\theta-\theta')$$

Now install our ring charge

$$\rho(r',\theta') = \rho_0\delta(r'-a)$$

to get

$$\varphi(\mathbf{x}) = \rho_0(a/2\pi\varepsilon_0) \int d\theta'\ (1/2)\ln(1/R^2) \qquad R^2 = r^2+a^2 - 2ra\cos(\theta-\theta')$$

$$= -\rho_0(a/4\pi\varepsilon_0) \int_{-\pi}^{\pi} dx\ \ln[r^2+a^2 - 2ra\cos x\ ]$$

$$= -\rho_0(a/2\pi\varepsilon_0) \int_{0}^{\pi} dx\ \ln[r^2+a^2 - 2ra\cos x\ ]$$

$$= -\rho_0(a/2\pi\varepsilon_0) \int_{0}^{\pi} dx\ \ln[(a^2)(1 + (r/a)^2 - 2(r/a)\cos x\ )]$$

$$= -\rho_0(a/2\pi\varepsilon_0) \int_{0}^{\pi} dx\ \{\ \ln(a^2) + \ln(1 + (r/a)^2 - 2(r/a)\cos x\ )\ \}$$

$$= -\rho_0(a/2\pi\varepsilon_0)[\ \ln(a^2)\pi + \int_{0}^{\pi} dx\ \ln(1 + (r/a)^2 - 2(r/a)\cos x\ )\ ]$$

I cannot find this second integral in GR7 where it ought to be. So I am now looking for

$$I \equiv \int_0^{\pi} dx \ \ln(1 + A^2 - 2A\cos x)]$$

Can Maple do this? It gives a big mess including lots of dilogs. But I remember this in **Stak and (6.22)** claims that

$$\ln(1 + A^2 - 2A\cos x) = -2 \Sigma_{n=1}^{\infty} (A^n/n)\cos(nx) \qquad // \text{ where } |A| < 1$$

Then my integral becomes

$$I \equiv \int_0^{\pi} dx \ \{-2 \Sigma_{n=1}^{\infty} (A^n/n)\cos(nx)\}$$

$$= -2 \Sigma_{n=1}^{\infty} (A^n/n) \int_0^{\pi} dx \cos(nx)$$

$$= -2 \Sigma_{n=1}^{\infty} (A^n/n) [\sin(n\pi)/n] \qquad // \text{ Maple}$$

But $\sin(n\pi) = 0$ for n = 1,2,3... and so this integral is I = 0. My answer is then

$$\varphi(\mathbf{x}) = - \rho_0(a/2\pi\varepsilon_0)[\ln(a^2)\pi] \qquad // \text{ inside}$$

which, if correct, says φ = a constant. This says that φ = this constant everywhere inside the ring of charge in 2D! I guess that is possible. Then for the outside region I will swap a and r and get

$$\varphi(\mathbf{x}) = - \rho_0(a/2\pi\varepsilon_0)[\ln(r^2)\pi] \qquad // \text{ outside}$$

Meanwhile, write $\rho_0 2\pi a = q$ for the total ring. Then

$$\rho_0(a/2\pi\varepsilon_0) = (q/2\pi a) (a/2\pi\varepsilon_0) = (q/4\pi^2\varepsilon_0)$$

and then our result is

$$\varphi(\mathbf{x}) = - (q/4\pi^2\varepsilon_0) [\ln(a^2)\pi] \qquad // \text{ inside}$$
$$\varphi(\mathbf{x}) = - (q/4\pi^2\varepsilon_0) [\ln(r^2)\pi] \qquad // \text{ outside}$$

or

$$\varphi(\mathbf{x}) = - (q/2\pi\varepsilon_0) \ln(a) \qquad // \text{ inside} \qquad (***)$$
$$\varphi(\mathbf{x}) = - (q/2\pi\varepsilon_0) \ln(r) \qquad // \text{ outside}$$

At least things match at the boundary.

If I take a→0 so the ring becomes a point charge, the result is

$\varphi(\mathbf{x}) = -(q/2\pi\varepsilon_0)\ln(r)$

and this agrees with my starting form (*) for such a point charge, so that is encouraging.

Here is a Maple plot with a = 2 and I omit the leading factor but keep the minus sign:

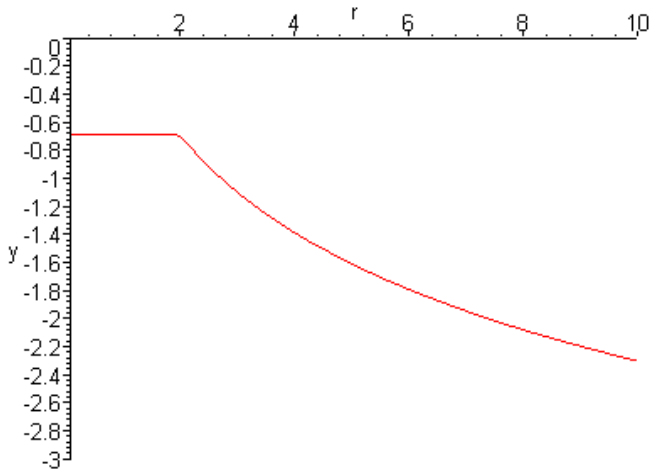
```
f := -piecewise(r<a,log(a),r>=1,log(r));
```

$$f := -\begin{cases} \ln(2) & r < 2 \\ \ln(r) & 1 \le r \end{cases}$$

```
a := 2;
```

$$a := 2$$

```
plot(f,r = 0.1..10, y=0..-3);
```



Recall that in 3D, inside a sphere of uniform charge you get φ = constant, and outside the sphere the charge appears as if it were q at the sphere center.

Here, in 2D we find that inside a circle of uniform charge we also get φ = constant. Outside the ring, the ring behaves as a point charge at the center of charge q. So very similar result.

Let's now go back and look at the integral again. I will do the integral and then we don't need that Stak 6.22 expansion.

$$I \equiv \int_0^{\pi} dx \ \ln(1 + \alpha^2 - 2\alpha \cos x)$$

Then

$$\partial_\alpha I = \int_0^{\pi} dx \ \frac{1}{1 + \alpha^2 - 2\alpha \cos x} (2\alpha - 2\cos x) = 2 \int_0^{\pi} dx \ \frac{\alpha - \cos x}{1 + \alpha^2 - 2\alpha \cos x}$$

$$= 2\alpha \int_0^{\pi} dx \frac{1}{1+\alpha^2 - 2\alpha \cos x} - 2 \int_0^{\pi} dx \frac{\cos x}{1+\alpha^2 - 2\alpha \cos x}$$

I would probably make this a contour integral, but let's try to look it up in GR7. The second integral seems to be this

2.6 $$\int_0^{\pi} \frac{\cos nx\, dx}{1 - 2a\cos x + a^2} = \frac{\pi a^n}{1-a^2} \qquad [a^2 < 1, \quad n \geq 0]$$
$$= \frac{\pi}{(a^2-1)\, a^n} \qquad [a^2 > 1, \quad n \geq 0]$$

so with n = 1 and a = α we get

$$\int_0^{\pi} dx \frac{\cos x}{1+\alpha^2 - 2\alpha \cos x} = \pi\alpha/(1-\alpha^2) \qquad \alpha < 1$$
$$= -(\pi/\alpha)/(1-\alpha^2) \qquad \alpha > 1$$

$$\int_0^{\pi} dx \frac{1}{1+\alpha^2 - 2\alpha \cos x} = \pi/(1-\alpha^2) \qquad \alpha < 1$$
$$= -(\pi)/(1-\alpha^2) \qquad \alpha > 1$$

So we then find

$$\partial_\alpha I = 1/(1-\alpha^2)\ \{\ 2\alpha\ \pi - 2\ \pi\alpha\ \} = 0 \qquad \alpha < 1$$

$$\partial_\alpha I = -1/(1-\alpha^2)\ \{\ 2\alpha\ \pi - 2\ \pi/\alpha\} \qquad \alpha > 1$$

This second is

$$\partial_\alpha I = -2\pi/(1-\alpha^2)\ \{\ \alpha\ - 1/\alpha\} = -2\pi/(1-\alpha^2)^*\ (\alpha^2 - 1)/\alpha = 2\pi/\alpha$$

So we summarize

$$\partial_\alpha I = 0 \qquad \alpha < 1$$
$$\partial_\alpha I = 2\pi/\alpha \qquad \alpha > 1$$

Now integrate to get

$$I = C_1 \qquad \alpha < 1$$
$$I = 2\pi \ln\alpha + C_2 \qquad \alpha > 1$$

Now match at α = 1 to get

$$C_1 = 2\pi \ln 1 + C_2 = C_2$$

so we then have

$I = C_1 \qquad \alpha < 1$
$I = 2\pi \ln\alpha + C_1 \qquad \alpha > 1$

where recall

$$I \equiv \int_0^{\pi} dx \ \ln(1 + \alpha^2 - 2\alpha \cos x)$$

How do we get the constant C_1? If α is very large we get

$$I = \int_0^{\pi} dx \ln(\alpha^2) = \pi \ln\alpha^2 = 2\pi\ln\alpha \qquad \text{large } \alpha$$

Equate this to

$$I = 2\pi \ln\alpha + C_1$$

to find that $C_1 = 0$. Thus

$I = 0 \qquad \alpha < 1$
$I = 2\pi \ln\alpha \qquad \alpha > 1$

or

$I = 2\pi\ln\alpha \ \theta(\alpha>1)$ // here is the integral all done!

Now go back to

$$\varphi(\mathbf{x}) = -\rho_0(a/2\pi\varepsilon_0)\left[\ln(a^2)\pi + \int_0^{\pi} dx \ \ln(1 + (r/a)^2 - 2(r/a)\cos x) \right]$$

$$= -\rho_0(a/2\pi\varepsilon_0)\left[\ln(a^2)\pi + I(\alpha = r/a)\right]$$

$$= -\rho_0(a/2\pi\varepsilon_0)\left[\ln(a^2)\pi + 2\pi\ln(r/a)\ \theta((r/a)>1)\right]$$

$$= -\rho_0(a/2\pi\varepsilon_0)\left[2\,\pi \ln(a) + 2\pi\ln(r/a)\ \theta(r>a)\right]$$

If $r < a$, the result is just

$$\varphi(\mathbf{x}) = -\rho_0(a/2\pi\varepsilon_0)\left[2\,\pi \ln(a)\right] \qquad r < a$$

If $r > a$ then

$$\varphi(\mathbf{x}) = - \rho_0(a/2\pi\varepsilon_0) [\ 2\ \pi \ln(a) + 2\pi\ln(r/a)\ ] = - \rho_0(a/2\pi\varepsilon_0)\ 2\pi\ln(r)$$

so then solution is then

$$\varphi(x) = - \rho_0(a/2\pi\varepsilon_0)\ 2\pi \ \{ \text{either } \ln(a) \text{ or } \ln(r) \}$$

Then recall

$$\rho_0(a/2\pi\varepsilon_0) = (q/2\pi a)\ (a/2\pi\varepsilon_0) = (q/4\pi^2\varepsilon_0)$$

so get

$$\varphi(x) = - (q/4\pi^2\varepsilon_0)\ 2\pi \ \{ \text{either } \ln(a) \text{ or } \ln(r) \}$$

$$= - (q/2\pi\varepsilon_0) \{ \text{either } \ln(a) \text{ or } \ln(r) \}$$

which then replicates our solution (***) above.

This solution also applies to an infinite cylinder with a uniform surface charge! I think I am going to make use of this fact when I look at the curl **J** around a cylinder in which **J** = constant!