Appendix M rewrite done Aug 19, 2014.

This version has a major error: you cannot assume that the Bessel function groups have "the same scale" when you go to low ω. As shown in version 2, at low ω it is the factor $|k/a^{-1}|$ that you want to show is small, not the factor $|k/\beta'|$.

Notes on Appendix M (appendix rewrite follows) **1**

Appendix M: Why the transverse vector potential A_t is small for a transmission line **9**

Notes on Appendix M (appendix rewrite follows)

I realized today that this Appendix refers to my now-defunct loss model and its low frequency cutoff concept. I now do things with k(ω) and I know that,

Fact 3: In the low frequency limit with G > 0 : (Q.3)

$$\text{Re}(k) \approx (\omega/2)\frac{RC+GL}{\sqrt{RG}}$$

$$\text{Im}(k) \approx -\sqrt{RG} - (\omega^2/8)\frac{(RC-LG)^2}{(RC)^{3/2}} \qquad \omega << R/L \text{ and } \omega << G/C$$

Fact 4: In the low frequency limit with G = 0 , (Q.4)

$$\text{Re}(k) \approx \sqrt{RC/2}\ \omega^{1/2} + \sqrt{C/8R}\ \omega^{3/2}$$

$$\text{Im}(k) \approx -\sqrt{RC/2}\ \omega^{1/2} + \sqrt{C/8R}\ \omega^{3/2} \qquad \omega << R/L$$

For G > 0, there is a lower finite limit for Im(k), but there is none for G = 0.
In both cases, Re(k) → 0.

So I will try to edit right in place here [below in this doc] . Red text is stuff that needs work! I will comment in this section with the edited result below.

The equation (1.5.9) really does have β_d sitting in it because this comes from the Helm equation in the dielectric. No matter what value k(ω) has at some ω, we still have β_d sitting in (1.5.9) as quoted below.

"Ouch, I claim that $\mathbf{J}_{t,i}(x,y,z) = e^{-j\beta_d z}\ \mathbf{J}_{t,i}(x,y)$ with β_d there instead of k. Did I do that back in Chapter 4? I am now looking thru Ch 4. The $e^{-j\beta_d z}$ gets into the discussion thru (1.5.9), yes. I then do the TL limit idea to get (4.3.7). But then bang, I say q(z) = q(0) $e^{-j\beta_d z}$ so everything is suddenly limited to the lossless theory! I am claiming that β_d is very small as a general rule, based on long λ idea. "

OK, I have now repaired this problem with Chapter 4, see notes in the normal edit log. [ok] I am back to Appendix M again below! // I have worked my way down to (M.6), but at that point big changes are going to be needed since I have k in place of β_d . I am trying to carry out Appendix M for the lossy case, not just for the lossless case where $k = \beta_d$. I need some general statement about the ratio

$$\text{ratio} = \frac{|k|}{|\beta'|}$$

I have now done lots of gyrations and I end up with

$$\left|\frac{k}{\beta}\right| < 10^{-8} \qquad \text{for all } \omega < 10^{18}$$

which is much different from my former claim of 10^{-3}. Do I believe this result? Let's check it at the extremes:

High ω: $k = \omega\sqrt{LC} - \frac{(RC + GL)}{2\sqrt{LC}}$ says (Q.2)

$= \omega/v_d - (RC+GL)\,(v_d/2)$

In my Belden example $R(\omega) = 4.8 \times 10^{-5}\sqrt{\omega}$ and $G = G = \tan_L \omega C$ so this becomes

$k = \omega/v_d - (4.8 \times 10^{-5}\sqrt{\omega}\, C + \tan_L \omega C\, L)\,(v_d/2)$

$= \omega/v_d - (4.8 \times 10^{-5}\sqrt{\omega} + \tan_L \omega\, L)\,(Cv_d/2)$

$= \omega/v_d - (4.8 \times 10^{-5}\sqrt{\omega}\,(Cv_d/2) + \tan_L \omega\, L\,(Cv_d/2)$

$= \omega/v_d - (4.8 \times 10^{-5}\sqrt{\omega}\,(Cv_d/2) + \tan_L \omega\,(v_d/2)\, LC$

$= \omega/v_d - (4.8 \times 10^{-5}\sqrt{\omega}\,(Cv_d/2) + \tan_L \omega\,(1/2v_d)$

$= (\omega/v_d)(1 + \tan_L/2) - (4.8 \times 10^{-5}\sqrt{\omega}\,(Cv_d/2)$

$\approx (\omega/v_d) - (4.8 \times 10^{-5}\sqrt{\omega}\,(Cv_d/2)$

$\approx (\omega/v_d)$

Ouch! I just realized that Appendix Q assumes R,L,G,C are all constants independent of ω! That means that in practice, Appendix Q is useless! I have to go now and deal with that, since it might have many implications!

9/3/14. Several weeks has now past, EV came and went, Appendix Q has been fully rewritten but not yet installed. I want now to continue these notes using the new App Q results! I am looking for the set of ω

such that |k| << |β|, since this is the defining condition for ignoring transverse action as noted below. So I want then to use my new "model" for |k|.

At high ω the new App Q (Q.3.5) shows that

$$\text{Re}(k) \approx (\omega/v_d) + (v_d\kappa C/2)\sqrt{\omega} + (1/4v_d)(\omega_d + v_d^4 C^2\kappa^2/2)\tan_L + O(1/\sqrt{\omega})$$
$$\text{Im}(k) \approx -(\omega/v_d)\tan_L/2 - (v_d\kappa C/2)\sqrt{\omega} - (1/2v_d)(\omega_d - v_d^4 C^2\kappa^2/2) + O(1/\sqrt{\omega})$$

where $\kappa \equiv \left(\frac{1}{p_1} + \frac{1}{p_2}\right)\sqrt{\frac{\mu}{2\sigma}}$ and $\omega_d \equiv (\sigma_d/\varepsilon_d)$

You can see that $|k| \approx |\text{Re}(k)| \approx (\omega/v_d)$ at high ω. This in this regime our inequality in question is this:

$|k^2| << |\beta|^2$?

$(\omega/v_d)^2 << \mu\sigma\omega$?

$\omega << \mu\sigma v_d^2$? dim(RHS) = hen/m * mho/m * m^2/sec^2 = 1/sec OK

RHS = $4\pi \times 10^{-7} *$ $5.81 \times 10^7 * 9 \times 10^{16}$ = $4\pi * 5.81 * 9 * 10^{16} = 113 \times 10^{16} \sim 10^{18}$

This is just my "usual calculation" and we then have |k| << |β| for all ω in the feasible high ω regime.

What about in the very low ω regime? There I show in new App Q that for $\omega_d = 0$,

Fact 4: The small ω limit for k(ω), assuming $\omega_d = 0$, is given by (Q.4.9)

$$\text{Re}(k) \approx +\sqrt{R_{dc}C/2}\ \sqrt{\omega}\,(1 - \tan_L/2) + O(\omega^{3/2})$$
$$\text{Im}(k) \approx -\sqrt{R_{dc}C/2}\ \sqrt{\omega}\,(1 + \tan_L/2) + O(\omega^{3/2})$$

So here we have roughly $|k|^2 = [\sqrt{R_{dc}C/2}\ \sqrt{\omega}\,]^2$ + same = $2\ R_{dc}C/2\ \omega = R_{dc}C\ \omega$, so then the question becomes this:

$|k^2| << |\beta|^2$?

$R_{dc}C\ \omega << \mu\sigma\omega$?

$R_{dc}C << \mu\sigma$?

$R_{dc}C << 73\ sec/m^2$?

This is a bit new I think. For the Belden cable we had in App R,

$R_{dc} = .036$ ohm/m $C = 69 \times 10^{-12}$ far/m

$R_{dc}C \sim 2.5 \times 10^{-12}$ which is <<<<< 73 !

Now look at the other case

Fact 3: The small ω limit for k(ω), assuming $\omega_d > 0$, is given by (Q.4.6)

$$\mathrm{Re}(k) \approx (\omega/2)\,(R_{dc} + \omega_d L_{dc})\,\sqrt{C/(\omega_d R_{dc})} + O(\omega^2) \qquad \omega < \omega_d = (\sigma_d/\varepsilon_d)$$
$$\mathrm{Im}(k) \approx -\sqrt{R_{dc}C\omega_d}\,\left[1 + (\tan_L/2)\,(\omega/\omega_d)\right] + O(\omega^2)$$

In this case, things are more complicated. We have

$$|k|^2 = \left[(\omega/2)\,(R_{dc} + \omega_d L_{dc})\,\sqrt{C/(\omega_d R_{dc})}\,\right]^2 + \left[-\sqrt{R_{dc}C\omega_d}\,\left[1 + (\tan_L/2)\,(\omega/\omega_d)\right]\right]^2$$

$$= (\omega/2)^2 (R_{dc} + \omega_d L_{dc})^2\, C/(\omega_d R_{dc}) + (R_{dc}C\omega_d)$$

$$\approx (R_{dc}C\omega_d) \text{ at very low } \omega$$

Then the question becomes:

$|k^2| << |\beta|^2$?

$(R_{dc}C\omega_d) << \mu\sigma\omega$?

At ω= 0 this is of course not true since RHS = 0. Write as

$R_{dc}C << \mu\sigma\,(\omega/\omega_d)$?

$2.5 \times 10^{-14} << (\omega/\omega_d)$?

But with $\omega_d = 113$ for Belden, this says we just need $\omega >> 10^{-12}$Hz which seems not much of a problem. So in any practical very low ω range we also have $|k^2| << |\beta|^2$!

I suspect that |k| << |β| for all ω, but don't have a proof. In "Q2 hi omega k version 3.mws" I arrive at this point,

```
aa^4;
```

$$\left(\kappa^2 w + w^2\left(\frac{1}{vd^2 C}+\frac{\kappa}{\sqrt{w}}\right)^2\right)\left(C^2 wd^2 + w^2 C^2\right)$$

```
collect(%,w);
```

$$\frac{w^4}{vd^4}+2\frac{\kappa C w^{\left(\frac{7}{2}\right)}}{vd^2}+2\kappa^2 C^2 w^3+\frac{wd^2 w^2}{vd^4}+2\frac{\kappa C wd^2 w^{\left(\frac{3}{2}\right)}}{vd^2}+2\kappa^2 C^2 wd^2 w$$

```
absbeta4 := mu^2*sigma^2*w^2;
```

$$absbeta4 := \mu^2 \sigma^2 w^2$$

The question is this: is aa^4 always $<< (\mu\sigma\omega)^2$ for all ω? I have just shown it is true for large ω. That means I have shown that

$\omega^4/v_d^4 << (\mu\sigma\omega)^2$ for all practical ω up to very high. perhaps 1 MHz to 10^9 GHz

For lower ω, you would have to show that the other terms in the "polynomial" above don't invalidate this inequality. I could plot both sides using the Belden 8281 specific parameters and find out. I guess that will be tomorrows task.

OK, I duly did the plot for Belden and it really does show that $|k| << |\beta|$ for all ω, from 0 Hz up to about 10^{16}Hz. So this is good news really, I never knew it before. The question now is: how would you show this is true analytically? I think you have to take the High and Low models separately and study each one:

<u>High ω Model</u>

$G(\omega) = C\,(\omega_d + \tan_L \omega)$ $\qquad \omega_d \equiv (\sigma_d/\varepsilon_d)$

$R(\omega) = \kappa\sqrt{\omega}$

$L(\omega) = 1/(Cv_d^2) + (\kappa/\sqrt{\omega})$. $\qquad \kappa \equiv \left(\frac{1}{p_1}+\frac{1}{p_2}\right)\sqrt{\frac{\mu}{2\sigma}}$ (Q.3.1)

$$k^2 = -(R+j\omega L)(G+j\omega C) = -(\kappa\sqrt{\omega}+j\omega[1/(Cv_d^2)+(\kappa/\sqrt{\omega})])(\omega_d+\tan_L\omega+j\omega)\,C$$

$$\approx -(\kappa\sqrt{\omega}+j\omega[1/(Cv_d^2)+(\kappa/\sqrt{\omega})])(\omega_d+j\omega)\,C$$

$$\approx -(\kappa\sqrt{\omega}+j\omega/(Cv_d^2)+j\,\kappa\sqrt{\omega})\;j\omega\,C$$

$$\approx -[(1+j)\,\kappa\sqrt{\omega}+j\omega/(Cv_d^2)]\;j\omega\,C$$

$$|k|^2 = \omega C\,|\,(1+j)\,\kappa\sqrt{\omega}+j\omega/(Cv_d^2)\,|$$

We want to compare this to

$|\beta|^2 = \mu\sigma\omega$

The comparison is then this:

$|k|^2 \ll |\beta|^2$??

$\omega C \left| (1+j)\,\kappa\sqrt{\omega} + j\omega/(Cv_d^2) \right| \ll \mu\sigma\omega$?

$C \left| (1+j)\,\kappa\sqrt{\omega} + j\omega/(Cv_d^2) \right| \ll \mu\sigma = 73$?

$1/v_d^2 = \varepsilon_d\mu_d$ so write this as

$C \left| (1+j)\,\kappa\sqrt{\omega} + j\omega\,\varepsilon_d\mu_d/C \right| \ll \mu\sigma = 73$?

$\left| (1+j)\,\kappa C\sqrt{\omega} + j\omega\,\varepsilon_d\mu_d \right| \ll \mu\sigma = 73$?

Well. really need to get the re and im parts, so write as

$\left| \kappa C\sqrt{\omega} + j\,(\omega\,\varepsilon_d\mu_d + \kappa C\sqrt{\omega}) \right| \ll \mu\sigma = 73$?

$\left| \kappa C\sqrt{\omega} + j\,(\omega\,\varepsilon_d\mu_d + \kappa C\sqrt{\omega}) \right|^2 \ll 73^2$?

$\kappa^2C^2\omega + (\omega\,\varepsilon_d\mu_d + \kappa C\sqrt{\omega})^2 \ll 73^2$

$2\kappa^2C^2\omega + (\varepsilon_d\mu_d)^2\omega^2 + 2\varepsilon_d\mu_d\,\kappa C\,\omega^{3/2} \ll 73^2$

$(\varepsilon_d\mu_d)^2\omega^2 + 2\varepsilon_d\mu_d\,\kappa C\,\omega^{3/2} + 2\kappa^2C^2\omega \ll 73^2$

NOW we can look at the coefficients for the Belden 8281 case:

```
(ed*mu0)^2;
```

$.6542774343\ 10^{-33}$

```
(evalf(kappa*C))^2;;
```

$.1124670917\ 10^{-28}$

```
evalf(2*ed*mu0*kappa*C);
```

$.1715630265\ 10^{-30}$

Then we ask

$.65 \times 10^{-33}\ \omega^2 + .17 \times 10^{-30}\ \omega^{3/2} + .22 \times 10^{-28}\ \omega \ll 73^2$?

$10^{-28}(.65 \times 10^{-5}\ \omega^2 + .17 \times 10^{-2}\ \omega^{3/2} + .22\omega) \quad << \quad 73^2 \quad ?$

Now the quantity in (...) is monotonic increasing. If we set $\omega = 10^{12}$ we get

$10^{-28}(.65 \times 10^{-5}\ 10^{24} + .17 \times 10^{-2}\ 10^{18} + .22\ 10^{12}) \quad << \quad 73^2 \quad ?$

$10^{-28}(.65 \times 10^{19} + .17 \times 10^{16} + .22 \times 10^{12}) \quad << \quad 73^2 \quad ?$

$10^{-28}\ 10^{12}(.65 \times 10^{7} + .17 \times 10^{4} + .22) \quad << \quad 73^2 \quad ?$

$10^{-16}(.65 \times 10^{7} + .17 \times 10^{4} + .22) \quad << \quad 73^2 \quad ?$

$.65 \times 10^{-9} \quad << \quad 73^2 \quad ?$

$6.5 \times 10^{-10} \quad << \quad 73^2 \quad ?$

$2.5 \times 10^{-5} << 73 \quad ?$ // this is $|k|^2 << |\beta|^2$?

$25 \times 10^{-6} << 73 \quad ?$

$5 \times 10^{-3} << 8.5 \quad ?$ // this is $|k| << |\beta|$?

$.001 << 1.7 \quad ?$

So this is in fact true for the Belden model at high ω.

Low ω Model

$G(\omega) = C (\omega_d + \tan_L \omega)$ $\qquad \omega_d \equiv (\sigma_d/\varepsilon_d)$

$R(\omega) = R_{dc}$

$L(\omega) = L_e + L_{idc} \equiv L_{dc}$. (Q.4.1)

$k^2 = - (R+j\omega L)(G+j\omega C) = -(R_{dc} + j\omega L_{dc})(\omega_d + \tan_L\omega + j\omega) C$

$\approx -(R_{dc} + j\omega L_{dc})(\omega_d + j\omega) C$

$-k^2 = C (R_{dc} + j\omega L_{dc})(\omega_d + j\omega) = C [(R_{dc} \omega_d - \omega^2 L_{dc}) + j \omega (L_{dc}\omega_d + R_{dc})]$

$|k|^4 = C^2 [(R_{dc} \omega_d - \omega^2 L_{dc})^2 + \omega^2 (L_{dc}\omega_d + R_{dc})^2]$

$|k|^4 = C^2 [(R_{dc} \omega_d - \omega^2 L_{dc})^2 + \omega^2 (L_{dc}\omega_d + R_{dc})^2]$

Then our question is

$|k|^4 \ll |\beta|^4$?

$C^2 [(R_{dc}\,\omega_d - \omega^2 L_{dc})^2 + \omega^2 (L_{dc}\omega_d + R_{dc})^2] \ll (\mu\sigma\omega)^2$

$C^2 [(R_{dc}\,\omega_d - \omega^2 L_{dc})^2 + \omega^2 (L_{dc}\omega_d + R_{dc})^2] \ll (\mu\sigma\omega)^2$

It is true that at $\omega = 0$ this inequality is not valid, but that is only below $\omega = 10^{-18}$ Hz !! Studying the above is not a trivial problem. LHS = $A\omega^4 + B\,\omega^2 + C$.

I think it is better just to do my plot rather than all the above analysis!

blue means "needs editing"

Appendix M: Why the transverse vector potential A_t is small for a transmission line

Claim: In the King gauge, the transverse vector potential $\mathbf{A_t}$ may be neglected for frequencies in the range 0 to 500 GHz. (M.1)

Overview: The vector potential is written below as $\mathbf{A_t} = \int_{\mathbf{conductors}} dxdy\ \mathbf{J_t}(x,y) * (stuff)$.

We shall make the following three claims:

(1) $|\mathbf{J_t}| < 10^{-3} |J_z|$ for f = 0 to 500 GHz (which is to say: "transverse currents are small inside the conductors") *(M.12)*

(2) in the $\mathbf{A_t}$ integral there is a cancellation effect not present in the A_z integral which in effect reduces $\mathbf{A_t}$ by a factor of 10 (ballpark) relative to A_z . *(M.14)*

(3) the net ballpark result is that $|\mathbf{A_t}| < 10^{-4} |A_z|$ for f = 0 to 500 GHz which is essentially the opening claim (M.1) above. *(M.22)*

As shown in the Anomaly section below, there is an exception to the Claim (M.1)

According to (1.5.9) one can express the King gauge vector potential at all points in space in terms of the currents in the transmission line conductors in this manner,

$$\mathbf{A}(\mathbf{x},\omega) = \frac{1}{4\pi} \Sigma_i \int \mu_i\ \mathbf{J_i}(\mathbf{x}',\omega) \frac{e^{-j\beta_d R}}{R} dV' \qquad R = |\mathbf{x} - \mathbf{x}'| \qquad (1.5.9)$$

where Σ_i is a sum over all the conductors, and β_d is the wavenumber in the dielectric. Therefore, the transverse part $\mathbf{A_t}$ may be written as an integral of the transverse conductor currents $\mathbf{J_{t,i}}$:

$$\mathbf{A_t}(\mathbf{x}) = \frac{1}{4\pi} \Sigma_i \mu_i \int \mathbf{J_{t,i}}(x',y',z') \frac{e^{-j\beta_d R}}{R} dx'\, dy'\, dz' \quad . \qquad (M.2)$$

Transverse refers to the x and y directions, where the infinite conductors are aligned in the z direction. The main current in a transmission line conductor is the longitudinal one J_z . When the above equation is processed in the manner of Chapter 4, and one assumes the transmission line limit, the result is

$$\mathbf{A_t}(\mathbf{x}) = -\frac{1}{4\pi} \Sigma_i \mu_i \int \mathbf{J_{t,i}}(x',y') \ln(s^2)\, dx'\, dy' \quad . \qquad s^2 = (x-x')^2 + (y-y'^2) \qquad (M.3)$$

As discussed in Section 4.3, in the transmission line limit both β_d and k are small (long wavelength), so $e^{-j\beta_d R} \approx 1$ and $e^{-jkR} \approx 1$. We assume the standard wave functional form such that $\mathbf{J_{t,i}}(x,y,z) = e^{-jkz}\ \mathbf{J_{t,i}}(x,y)$ and then set $e^{-jkz} \approx 1$ for the contributing portion of the dz' integration and that integration produces $-\ln(s^2)$ as in Ch 4 or (J.10).

Meanwhile, Appendix D computes the **E** fields inside a round conductor for each partial wave m, and here we multiply them each by σ to get the current density components,

Current Densities in a Round Wire: $R_{dc} = \frac{1}{\sigma\pi a^2}$ $\beta'^2 = \beta^2 - k^2$ *(D.2.33)*

$J_z(r,m) = \sigma(1/4)\,\eta_m\, I\, R_{dc}\, (a\beta')\, f_m$ $f_m = [\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}]$ $x = \beta' r$

$J_r(r,m) = \sigma(j/4)\,\eta_m\, I\, R_{dc}\, (ak)\, g_m$ $g_m = [\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}]$ $x_a = \beta' a$

$J_\theta(r,m) = \sigma(1/4)\,\eta_m\, I\, R_{dc}\, (ak)\, h_m$ $h_m = [\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}]$

The currents are expressed in terms of a cylindrical coordinate system whose z axis runs down the center of the round conductor. Coefficient η_m is the "surface charge moment" of the mth partial wave, and the θ-space currents are given by (D.1.3a),

$$\mathbf{J}(r,\theta) = \sum_{m=-\infty}^{\infty} \mathbf{J}(r,m)\, e^{jm\theta} \quad . \qquad \text{// partial wave expansion} \qquad (M.4)$$

The moments η_m may be obtained by solving the transmission line "capacitor problem" as outlined in Section 6.5 (a). One finds potential φ, then **E**, then surface charge n(θ), and finally η_m.

The current components are,

$$\mathbf{J} = J_z\hat{\mathbf{z}} + J_r\hat{\mathbf{r}} + J_\theta\hat{\boldsymbol{\theta}} = J_z\hat{\mathbf{z}} + \mathbf{J}_t \qquad \mathbf{J}_t = J_r\hat{\mathbf{r}} + J_\theta\hat{\boldsymbol{\theta}} \quad . \qquad (M.5)$$

Rather than study these round wire internal solutions in detail, we make two observations:

Observation (1): The transverse currents J_r and J_θ are very small compared to J_z. (M.6)

Looking at (D.2.33) above one sees that the transverse currents J_r and J_θ are in general *smaller* than the longitudinal current J_z by factor |k/β'|. In making this claim we regard the combinations of Bessel functions shown in (D.2.33) as being of the same general scale, which can be confirmed by doing plots of the various complex function magnitudes (for large ω).

So how small is |k/β'| ? Since $\beta'^2 = \beta^2 - k^2$ we may write

$$|k/\beta'|^2 = \frac{|k/\beta|^2}{1 - |k/\beta|^2} \quad . \qquad (M.7)$$

If we can show that $|k/\beta| = \alpha << 1$, then we know that $|k/\beta'| \approx |k/\beta| = \alpha << 1$. So we then ask: how small is $|k/\beta|$? Recall that

$$\beta^2 = - j\omega\mu\sigma \qquad \textit{(1.5.1d)} \qquad \text{(M.8)}$$

$$k = - j\sqrt{(R+j\omega L)(G+j\omega C)} \qquad \textit{(5.3.5)} \qquad \text{(M.9)}$$

so therefore

$$|k/\beta| = \sqrt{\frac{|(R+j\omega L)(G+j\omega C)|}{\omega\mu\sigma}} \quad . \qquad \text{(M.10)}$$

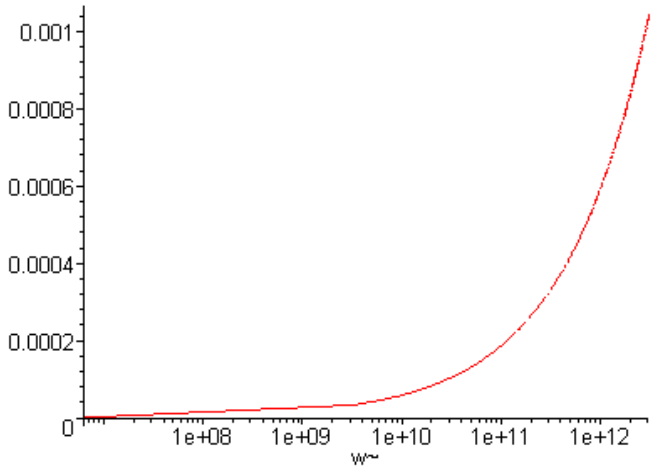
For "large" ω we find that [see (4.11.34) and following comments]

$$|k/\beta| = \sqrt{\frac{|(j\omega L)(j\omega C)|}{\omega\mu\sigma}} = \sqrt{\frac{\omega^2 LC}{\omega\mu\sigma}} \approx \sqrt{\frac{\omega^2 L_e C}{\omega\mu\sigma}} = \sqrt{\frac{\omega^2\mu_d\varepsilon_d}{\omega\mu\sigma}} = \sqrt{\frac{\omega\mu_d\varepsilon_d}{\mu\sigma}}$$

$$\approx \sqrt{\frac{\varepsilon_d\omega}{\sigma}} \approx \sqrt{\frac{2.3\varepsilon_0\omega}{\sigma}} \quad . \qquad \text{(M.11)}$$

We have assumed that $L = L_e$, $\mu_d = \mu = \mu_0$ and $\varepsilon_d/\varepsilon_0 = 2.3$. Here then is a plot of $|k/\beta|$ for $f = \omega/2\pi$ in the range 1 MHz to 500 GHz.

```
sigma := 5.81e7: e0 := 8.85e-12: pi = evalf(Pi):
semilogplot(sqrt(2.3*e0*w/sigma),w=2*pi*1e6..1*pi*1e12,numpoints = 1000);
```



This shows that in fact $|k/\beta|$ *is* small, being < .001. But what about "low" frequencies? To get a more complete view we may use the crude transmission line parameter model of Appendix Q,

Simple Transmission Line Parameter Model *(Q.1.9)*

$C(\omega) = C$ = independent of ω $\qquad \kappa \equiv \left(\frac{1}{p_1} + \frac{1}{p_2}\right)\sqrt{\frac{\mu}{2\sigma}}$

$G(\omega) = C\,(\omega_d + \tan_L \omega)$ $\qquad \omega_d \equiv (\sigma_d/\varepsilon_d)$ $\qquad G_{dc} \equiv \omega_d\, C$

$R(\omega) = R_{dc}\,\theta(\omega<\omega_R) + (\kappa\sqrt{\omega})\,\theta(\omega\geq\omega_R)$ $\qquad \omega_R \equiv (R_{dc}/\kappa)^2$

$L(\omega) = L_e + L_{idc}\,\theta(\omega<\omega_L) + (\kappa/\sqrt{\omega})\,\theta(\omega\geq\omega_L)$ $\qquad \omega_L \equiv (\kappa/\,L_{idc})^2$

$L_e C = \mu_d \varepsilon_d = 1/v_d^2$ $\qquad$ *(4.11.34)*

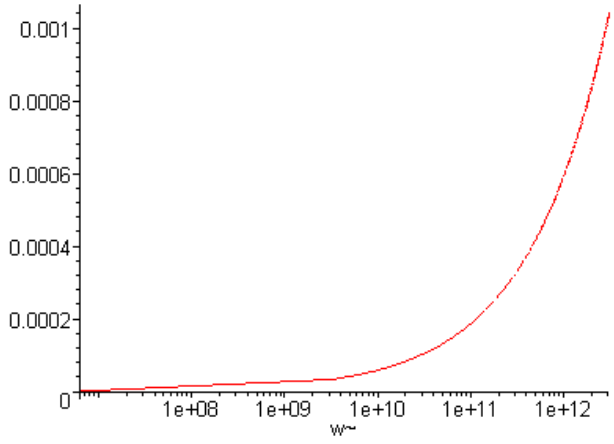
Using this model, we may again compute the ratio $|k/\beta|$ for all ω,

$$|k/\beta| = \sqrt{\frac{|(R+j\omega L)(G+j\omega C)|}{\omega\mu\sigma}} \quad . \qquad (M.9)$$

In passing, we note that the relation $k = -j\sqrt{(R+j\omega L)(G+j\omega C)}$ is accurate for the network model of a transmission line (Appendix K), but is questionable at low ω for the true physics model of a transmission line as presented in Chapters 4 and 5. We use the k expression nevertheless, with the presumption that it is roughly correct at low ω.

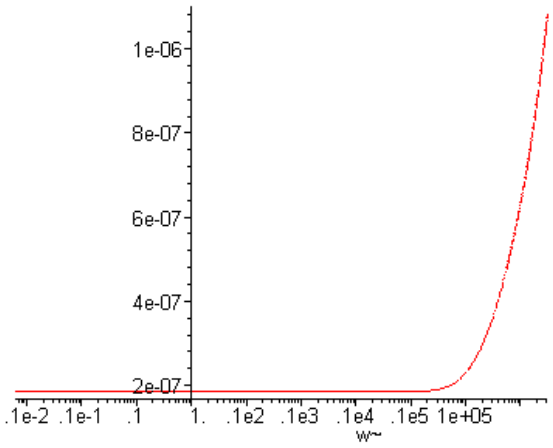
The $|k/\beta|$ ratio involves the transmission line parameters like R_{dc}, C, L_e, L_{idc} and σ_d in a complicated way. For parameters appropriate to the Belden 8281 coaxial cable of Appendix R, we can have Maple compute and then plot the ratio. Omitting the Maple details, we first show the resulting plot of $|k/\beta|$ for frequency in the range 1 MHz to 500 GHz :

```
semilogplot(absk/absbeta,w=2*pi*1e6..1*pi*1e12,numpoints=1000);
```



The plot is basically the same as the previous plot using the large ω limit of the ratio. For the range 10^{-4} Hz to 1 MHz we get instead

```
semilogplot(absk/absbeta,w=2*pi*1e-4..1*pi*1e6,numpoints=2000);
```



which says $|k/\beta| < 10^{-6}$ which is much smaller than the ratio at the higher end of the range. Although our model is inaccurate in the mid ω range, it seems reasonable to nevertheless arrive at this :

Conclusion: For frequencies from 0 to 500 GHz, $|k/\beta| < .001$ and therefore $|k/\beta'| < .001$ and therefore the ratio of transverse to longitudinal current densities is $< .001 = 10^{-3}$. That is to say

$$|J_r| < 10^{-3}\,|J_z|$$
$$|J_\theta| < 10^{-3}\,|J_z| \qquad f = 0 \text{ to } 500 \text{ GHz} \qquad \text{(M.12)}$$

The conclusion then is that the transverse currents are less than 1/1000th of the size of the longitudinal currents for the round copper conductor at all frequencies of interest below 500 GHz, and we can reasonably assume that a similar conclusion applies to a conductor of any cross sectional shape (while maintaining the transmission line limit). This then concludes our "proof" of the claim that "transverse currents are very small" inside the conductors of a transmission line.

Anomaly: For a conducting dielectric ($\sigma_d > 0 \Rightarrow \omega_d > 0$) the ratio $|k/\beta|$ for small ω has the form

$$|k/\beta| = \sqrt{\frac{|(R+j\omega L)(G+j\omega C)|}{\omega\mu\sigma}} \rightarrow \sqrt{\frac{|R_{dc}(C\omega_d+j\omega C)|}{\omega\mu\sigma}} = \sqrt{\frac{R_{dc}C\omega_d}{\omega\mu\sigma}} = \sqrt{\frac{R_{dc}G_{dc}}{\omega\mu\sigma}} \qquad \text{(M.13)}$$

which becomes infinite at ω = 0. The issue here is that as $\omega \rightarrow 0$ we have $\beta \rightarrow 0$ but $k \rightarrow -j\sqrt{R_{dc}G_{dc}} \neq 0$. Since the dielectric conducts current, there must be a DC loss going down the transmission line in z and this is associated with $\mathrm{Im}(k) = -\sqrt{R_{dc}G}$, so it is *expected* that $k \not\rightarrow 0$. We then expect our 10^{-3} inequality rule of (M.12) to be violated when

$$10^{-6} < R_{dc}G_{dc}/(\omega\mu\sigma)$$

or

$$\omega < 10^{6}\, R_{dc}G_{dc}/(\mu\sigma) \approx 10^{6}\, R_{dc}G_{dc}/73 \; . \qquad \text{// copper conductors}$$

For the Belden 8281 cable of Appendix R, we find that $R_{dc} \approx .036$, $G_{dc} = \omega_d C \approx 0.8 \times 10^{-14}$ so our condition for violation becomes

$$\omega < 10^6 (.036)(0.8 \times 10^{-14}/73) = 3.9 \times 10^{-12} \text{ sec}^{-1}$$

It is not unreasonable that at such ultra-low frequencies near DC the transverse current J_r can be large relative to J_z. The network model is then that of Fig K.4. One could solve the DC magnetostatics problem for an infinite transmission line with round conductors to determine J_r and J_z and compare their magnitudes. Since a practical transmission line is not operated at such ultra low frequencies, we ignore the anomaly and state the claim (M.1) as if it were valid all the way down to $\omega = 0$.

NOTE in red: In (D.11.14) I claim to know J_z and J_r as $\omega \to 0$ for each partial wave. The ratio seems to be this

$$J_r/J_z \approx aG / \sqrt{G/R} = a\sqrt{GR}$$

For Belden 8281 center wire this is

$$J_r/J_z = a_1 \sqrt{GR} = .00039 * \sqrt{(.036)(0.8 \times 10^{-14}/73)} = .7 \times 10^{-11}$$

which disagrees with the above discussion!! I do state there that $k \approx -j\sqrt{RG}$ which agrees with above. In box (D.11.7) the ratio E_r/E_z is clearly just (ak). What happened to (k/β) ? Aha! It is those Bessel functions that I ignored above !!!!

Observation (2): In the Helmholtz integration (M.3) there is a large amount of cancellation. (M.14)

Let us consider the nature of this integration in the illustrative case of a two round conductors,

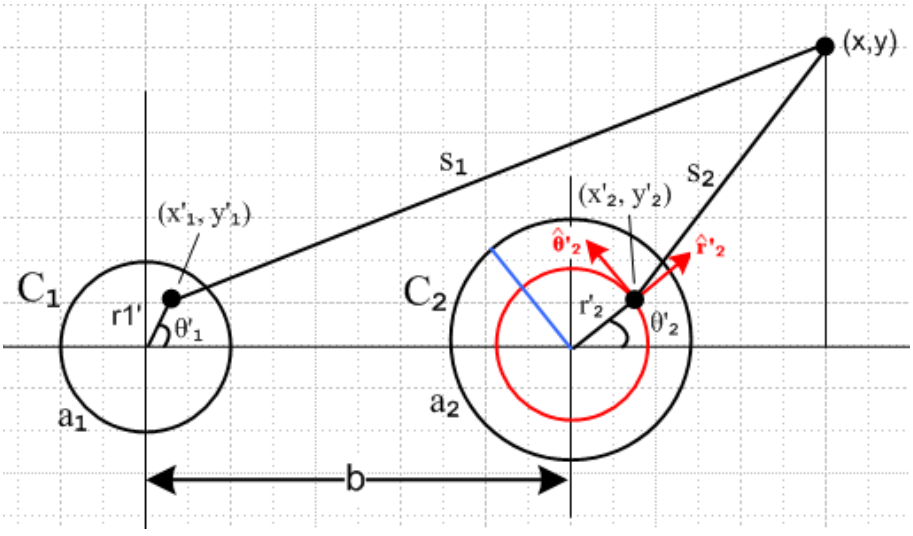


Fig M.1

Consider the contribution to the transverse vector potential component A_x from the right conductor C_2,

$$A_x(\mathbf{x}) = -\frac{\mu_d}{4\pi}\int J_x(x'_2,y'_2)\,\ln(s_2^2)\,dx'_2\,dy'_2 \qquad s_2^2 = (x-x'_2)^2 + (y-y'_2{}^2) \tag{M.15}$$

or

$$A_x(\mathbf{x}) = -\frac{\mu_d}{4\pi}\int \left[J_r(r'_2,\theta'_2)\,\hat{\mathbf{r}}'_2 \bullet \hat{\mathbf{x}} + J_\theta(r'_2,\theta'_2)\,\hat{\boldsymbol{\theta}}'_2 \bullet \hat{\mathbf{x}}\right]\,\ln(s_2^2)\,[a_2 d\theta'_2]\,dr'_2 \tag{M.16}$$

or

$$A_x(\mathbf{x}) = -\frac{\mu_d}{4\pi}\int \left[J_r(r'_2,\theta'_2)\cos\theta'_2 - J_\theta(r'_2,\theta'_2)\sin\theta'_2\right]\,\ln(s_2^2)\,[a_2 d\theta'_2]\,dr'_2\,. \tag{M.17}$$

The transverse currents in conductor C_2 have this partial wave expansion from (M.4),

$$J_r(r'_2,\theta'_2) = \sum_{m=-\infty}^{\infty} J_r(r'_2,m)\,e^{jm\theta'_2} \tag{M.18}$$

and similarly for J_θ . Thus we get

$$\begin{aligned} A_x(x,y) = &-\frac{\mu_d}{4\pi}\,a_2\,\Sigma_m\,\int_0^{a_2} dr'_2\,J_r(r'_2,m)\,\int_0^{2\pi} d\theta'_2\,e^{jm\theta'_2}\cos\theta'_2\,\ln(s_2^2) \\ &+\frac{\mu_d}{4\pi}\,a_2\,\Sigma_m\,\int_0^{a_2} dr'_2\,J_\theta(r'_2,m)\,\int_0^{2\pi} d\theta'_2\,e^{jm\theta'_2}\sin\theta'_2\,\ln(s_2^2) \end{aligned} \tag{M.19}$$

where, from (D.2.33) quoted above,

$$J_r(r,m) = \sigma(j/4)\,\eta_m\, I\, R_{dc}\ (a\beta_d)\ \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x = \beta' r$$

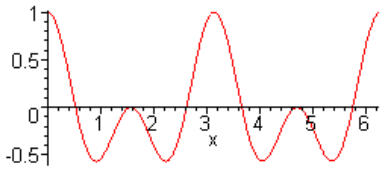
$$J_\theta(r,m) = \sigma(1/4)\,\eta_m\, I\, R_{dc}\ (a\beta_d)\ \left[\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}\right] \qquad x_a = \beta' a\ .$$

It is in theory possible to first do the $d\theta'_2$ integration in (M.19) and then do the dr'_2 integration and get an analytic result for $A_x(x,y)$. We have dealt with similar angle integrations elsewhere in this document. Rather then attempt this task, we instead consider the portion of the 2D integration represented by the red ring in Fig M.1. On this ring, r'_2 is constant, and our interest is the θ'_2 integration. For any value of m (except for ±1) the trigonometric functions like $e^{jm\theta'_2}\cos\theta'_2$ integrate to 0, for example,

```
f := cos(3*x)*cos(x);
```

$$f := \cos(3x)\cos(x)$$

```
plot(f,x = 0..2*Pi);
```




```
int(f,x = 0..2*Pi);
```

$$0$$

Fig M.2

For these values of m, were it not for the fact that s_2^2 varies around the red circle, $A_x(x,y)$ would be identically 0. Although s_2^2 does vary on the red circle, $\ln(s^2)$ varies very little, and we expect to still have this strong cancellation in the θ'_2 integral so $A_x(x,y)$ is then small. It is true that if $\mathbf{x}$ and $\mathbf{x}'_2$ were to approach the conductor boundary from opposite sides, then $\ln(s^2)$ would vary a lot more and the cancellation would be less, but we ignore this detail in our qualitative argument.

For m = ± 1 this smallness argument fails since for example $\cos^2(\theta'_2)$ does not average to 0 around the red ring. Ignoring the θ'_2 variation in s_2^2 we get in this case (setting $e^{\pm j\theta'_2} \sim \cos\theta'_2$)

$$A_x(x,y) \approx -\frac{\mu_d}{4\pi} a_2 \sum_m \int_0^{a_2} dr'_2\ J_r(r'_2,m)\ln(s_2^2) \int_0^{2\pi} d\theta'_2 \cos^2\theta'_2$$

$$\approx -\frac{\mu_d}{4\pi} a_2 \sum_m \int_0^{a_2} dr'_2\ J_r(r'_2,\pm 1)\ln(s_2^2)\,\pi \qquad \text{(M.20)}$$

Now we make a different argument which concerns the behavior of the complex Bessel functions as a function of r'_2. As studied in Chapter 2, these functions have a dramatically oscillating phase even in the soft skin depth limit, and we expect then to get cancellation due to this phase as we integrate on the radial segment shown blue in Fig M.1, and again $\ln(s_2^2)$ varies slowly on this ray due to the nature of ln.

The arguments made above for $A_x(x,y)$ also apply to $A_y(x,y)$, and it seems reasonable to assume that the arguments are generally valid for an arbitrary conductor cross section.

Admittedly our analysis here is imprecise and qualitative, but we think it is convincing that there is in fact much cancellation when the transverse currents are integrated over the conductors.

This stands in stark contrast to the longitudinal situation where A_z, being the Helmholtz integral of J_z, involves a generally non-cancelling integration (per conductor) over a generally large current component.

We now wish to compare the following two integrals, where we pick component A_r to represent a transverse component of $\mathbf{A}_t$,

$$A_r(\mathbf{x}) = -\frac{1}{4\pi} \Sigma_i \mu_i \int J_{r,i}(x',y') \ln(s^2)\, dx'\, dy' \quad . \qquad s^2 = (x-x')^2 + (y-y'^2) \qquad \textit{(M.3)}$$

$$A_z(\mathbf{x}) = -\frac{1}{4\pi} \Sigma_i \mu_i \int J_{z,i}(x',y') \ln(s^2)\, dx'\, dy' \quad . \qquad s^2 = (x-x')^2 + (y-y'^2) \qquad (M.21)$$

We have shown in (M.13) that $| J_r | < 10^{-3} | J_z |$. Without any mathematical rigor, and allowing a factor of 10 "gain" from the cancellation effect of Observation (2), we make the following *ballpark* estimate,

$$|\mathbf{A}_t| < 10^{-4} |A_z| \qquad f = 0 \text{ to } 500 \text{ GHz.} \qquad (M.22)$$

It is assumed that as ω increases, the transmission line geometry is appropriately shrunk so the transmission line limit remains operative.