Hall Effect for General B PhL 4.1.14

Reviewed 5/6/14. This is an important document. In Section 1 I reconsider the Hall effect for the B field in an arbitrary direction. But I put this development on hold while in Section 2 I study my existing use of regular Ohm's Law in lines doc Chapter 1 and Appendix D (the two places I guess I use it). I fear that all of lines doc is wrong because I failed to use the "magnetic Ohm's Law". But then in Section 3 I resume where I left off in Section 1 and I conclude that regular Ohm's Law is "in general" OK to use as long as the G field is less than 569 tesla ! I just wrote this all up as new Section N.8 and added it to lines doc!

1. Hall Effect with General B 1

2. Digression: Review of the Use of Ohm's Law in Chapter 1 and then Appendix D 2

3. Go back to Magnetic Ohm's Law again. We had for time-static situation that 3

1. Hall Effect with General B

I am here trying to keep the same Hall Effect picture as now in Appendix N, but B is in an arbitrary direction instead of just in the z direction.

Consider the Hall effect where we change from the earlier figure so now a rectangular conductor runs in the z direction and x and y are transverse directions, as in our transmission line analysis. In fact, let's make it very long relative to its transverse dimensions. What can we say about the Magnetic Ohm's Law in this situation?

We start with the following, using $\kappa \equiv 1/\mu$.

$$\kappa \mathbf{v} = \mathbf{E} + \mathbf{v} \times \mathbf{B} \qquad 1$$

where now B is an arbitrary applied B field. At any point inside the conductor, it has some magnitude and direction, so B is a completely general **B**(r), subject to the requirement that div **B** = 0 . If this external field comes from some physical source, we know that div **B** = 0 will be respected.

Then using 1,2,3 as r,θ,z we can say

$$\kappa v_i = E_i + \varepsilon_{ijk} v_j B_k$$

so

$$\kappa v_1 = E_1 + v_2 B_3 - v_3 B_2$$
$$\kappa v_2 = E_2 + v_3 B_1 - v_1 B_3$$
$$\kappa v_3 = E_3 + v_1 B_2 - v_2 B_1 \qquad 2$$

We know that $J_i = n q v_i$. We will be looking for a Maxwell-Equations solution inside the conductor which has $J_1 = 0$ and $J_2 = 0$, so no transverse currents at surface or interior. This means $v_1 = 0$ and $v_2 = 0$. In this case, the above equations simplify to these

$$0 = E_1 - v_3 B_2$$

$0 = E_2 + v_3 B_1$

$\kappa v_3 = E_3$

Notice that component B_3 plays no role in the solution. We then have

$E_1 = B_2 v_3$ $\quad\quad$ $J_1 = nqv_1 = 0$

$E_2 = -B_1 v_3$ $\quad\quad$ $J_2 = nqv_2 = 0$

$E_3 = \kappa v_3$ $\quad\quad$ $J_3 = nqv_3 = (nq\mu)E_3 = \sigma E_3$

- no current densities in the transverse directions 1 and 3
- normal Ohm's Law in the 3 direction
- Hall fields in both transverse directions, E_1 and E_2

We shall seek a solution in which v_3 = constant, to see if such a solution exists. We have already assumed that div **B** = 0 for the applied **B** field, so we have three other DC Maxwell equations to check out.

curl **B** = μ_0**J**

curl **E** = 0

div **E** = 0

Let's now make these definitions:

$\mathbf{B_1}$ = B field that would be generated by our conductor with current I in isolation

$\mathbf{B_2}$ = the applied external field

B = $\mathbf{B_1}$+ $\mathbf{B_2}$ = the total magnetic field.

We now want to face the idea of two superposed problems.

2. Digression: Review of the Use of Ohm's Law in Chapter 1 and then Appendix D

Motivation: When I was writing this, I was thinking I wrongly used "regular" Ohm's Law in various places in lines doc. I have since realized that regular Ohm's Law is OK up to something like 160 GHz. I first here scan Chapter 1 of lines doc, then Appendix D.

I am looking at where Ohm's Law is *used* in lines doc Chapter 1.

Section 1.1. It is stated and commented on. I guess I need to add some magnetic extra words in that comment! (but not yet) . This is comment 8. I see now further use of Ohm in the rest of this section.

Section 1.2 field wave equations -- NOT used, only Maxwell's are used

Section 1.3 potential wave equations.

(a) -- NOT used

(b) relativity, not used

(c) used for the first time just above (1.3.19). We are in the dielectric. Consider:

κ **v** = **E** + **v x B**

I can argue that the dielectric is only very weakly conducting, so v is extremely small, so this equation simplifies to $\kappa \mathbf{v} = \mathbf{E}$ and this gives the regular Ohm's Law which I use there. This is for region 1 which is the dielectric. For region 2 and region 3 I don't use Ohm's Law to eliminate the current, so these conclusions I think are OK:

$$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1\partial_t)\,\mathbf{A} = 0 \qquad \text{region 1}$$
$$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1\partial_t)\,\mathbf{A} = -\mu_2\mathbf{J}_2 \qquad \text{region 2}$$
$$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1\partial_t)\,\mathbf{A} = -\mu_3\mathbf{J}_3 \qquad \text{region 3} \qquad (1.3.21)$$

We combine these into a single equation which is then valid over all of region R,

$$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1)\,\mathbf{A} = -\mu_2\mathbf{J}_2 - \mu_3\mathbf{J}_3 \qquad \text{all of region R} \qquad (1.3.22)$$

Potential Wave Equations in the King Gauge (1.3.28)

$$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1\partial_t)\varphi = -(1/\varepsilon_1)\,\Sigma_{i=2}^{N+1}\rho_i \qquad \text{all of region R} \qquad \textit{(1.3.27)}$$
$$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1\partial_t)\mathbf{A} = -\Sigma_{i=2}^{N+1}\,\mu_i\mathbf{J}_i \qquad \text{all of region R} \qquad \textit{(1.3.23)}$$
$$\text{div}\,\mathbf{A} = -\mu_1\varepsilon_1\,\partial_t\varphi - \mu_1\sigma_1\varphi \qquad \text{King gauge} \qquad \textit{(1.3.18)}$$

So all of the above is OK on Ohm, given that small v in the dielectric.

But now we continue on toward the end of section (c) way down starting with (1.3.35). Here I am suddenly talking about **E** and **B** wave equations again. I think (1.3.35) is OK and Ohm has not been used yet, although I state Ohm right there.

But now (1.3.36) HAS used regular Ohm's Law, so we are getting into trouble here. Here I am using Ohm in all three regions! I am getting into trouble because when these E and B field equations go into the ω domain, we get into Appendix D and things are affected. But be patient.

Section 1.4 retarded solutions -- no mention of Ohm.

Section 1.5

(a) going now into the ω domain! The potential wave equations (1.5.3 etc are all OK because they come from (1.3.28) above which is OK. The small v in dielectric is saving us here.

(b) Helm integral stuff all OK, no Ohm used

(c) King leading factor stuff. Below Fig 1.7 I mention Ohm. Lots or σ stuff. I guess I have to appeal to the "low ω" argument to justify this section.

(d) here (1.5.25) and (1.5.27) are OK, but (1.5.27) has assumed general Ohm's Law. All else is OK then except the final (1.5.32) where we need the large ω argument to survive.

The rest of Chapter 1 is then OK.

Appendix D: Right off the bat I am using (1.3.36) which, as just noted above, assumes regular Ohm's Law. So all of Appendix D lies under this cloud and I have to clarify the "low ω" argument in order to save this appendix!

3. Go back to Magnetic Ohm's Law again. We had for time-static situation that

$\kappa v_1 = E_1 + v_2 B_3 - v_3 B_2$

$\kappa v_2 = E_2 + v_3 B_1 - v_1 B_3$

$\kappa v_3 = E_3 + v_1 B_2 - v_2 B_1$ 2

Maple solves this problem for the three v_i as follows: [this is in hall3.mws]

```
restart:
eq1 := kappa*v1 - B3*v2+B2*v3 = E1;
```

$$eq1 := \kappa\, v1 - B3\, v2 + B2\, v3 = E1$$

```
eq2 :=  kappa*v2 - B1*v3+B3*v1 = E2;
```

$$eq2 := \kappa\, v2 - B1\, v3 + B3\, v1 = E2$$

```
eq3 :=  kappa*v3 - B2*v1+B1*v2 = E3;
```

$$eq3 := \kappa\, v3 - B2\, v1 + B1\, v2 = E3$$

```
f := solve({eq1,eq2,eq3},{v1,v2,v3});
```

$$= \left\{ v3 = \frac{B2\,\kappa\,E1 - B1\,\kappa\,E2 + B1\,B3\,E1 + B2\,E2\,B3 + E3\,\kappa^2 + E3\,B3^2}{\kappa\,(B2^2 + B1^2 + \kappa^2 + B3^2)},\right.$$

$$v2 = \frac{\kappa^2\,E2 + \kappa\,B1\,E3 - E1\,B3\,\kappa + B1\,B2\,E1 + B2\,B3\,E3 + E2\,B2^2}{\kappa\,(B2^2 + B1^2 + \kappa^2 + B3^2)},$$

$$\left. v1 = \frac{E1\,\kappa^2 - \kappa\,B2\,E3 + E1\,B1^2 + B1\,B2\,E2 + B1\,E3\,B3 + \kappa\,E2\,B3}{\kappa\,(B2^2 + B1^2 + \kappa^2 + B3^2)} \right\}$$

Now how can I write this in terms of things like $\omega_c\tau$?? I did this in Appendix N for the simple Hall case. Recall that

$\omega_c\tau = \mu B = B/\kappa$ => $\omega_{ci}\tau = \mu B_i = B_i/\kappa$

so we have an ω_c for each direction and at each point in space $\omega_{ci}(\mathbf{r})$. In appendix N I just replaced the B field with the ω_c stuff, so let's do that same thing for each B_i component here. Well, if I do that for V1 I get this for J_3:

$$\frac{\sigma\,(\omega 2\,\tau\,E1 - \omega 1\,\tau\,E2 + \omega 1\,\tau^2\,\omega 3\,E1 + \omega 2\,\tau^2\,E2\,\omega 3 + E3 + E3\,\omega 3^2\,\tau^2)}{\omega 2^2\,\tau^2 + \omega 1^2\,\tau^2 + 1 + \omega 3^2\,\tau^2}$$

I will write this out:

num $J_3 = \sigma$ [$E_3 + (\omega_2\tau)E_1 - (\omega_1\tau)E_2 + (\omega_1\tau)(\omega_3\tau)E_1 + (\omega_2\tau)(\omega_3\tau)E_2 + (\omega_3\tau)^3 E_3$]

$= \sigma$ [$E_3\{1 + (\omega_3\tau)^3\} + E_1\{(\omega_2\tau) + (\omega_1\tau)(\omega_3\tau)\} + E_2\{-(\omega_1\tau) + (\omega_2\tau)(\omega_3\tau)\}$]

den $J_3 = 1 + (\omega_1\tau)^2 + (\omega_2\tau)^2 + (\omega_3\tau)^2$

So if we can assume that $\omega_i\tau << 1$ for all three components, then we just get

$$J_3 = \sigma E_3$$

and this is then the generalization of what happens in Appendix N. If I now set $\tau = 0$, I get the regular Ohm's law in all three directions, I just did this in Maple to make sure. This is the same as making all three B fields be very small, and then

$$\kappa \mathbf{v} = \mathbf{E} + \mathbf{v} \times \mathbf{B} \quad \Rightarrow \quad \kappa \mathbf{v} = \mathbf{E} \quad \Rightarrow \quad \mathbf{J} = \sigma \mathbf{E}$$

So what I really need is a condition that B be "small".

$$\omega_{ci}\tau = \mu B_i = B_i/\kappa << 1 \qquad B_i = \kappa (\omega_i \tau)$$

This then requires that

$$B_i << \kappa \qquad \kappa = 1/\mu \qquad \mu = (q\tau/m) \qquad \kappa = (m/q\tau)$$

so we need then

$$B_i << (m/q\tau)$$ // I will evaluate this below

This has nothing to do with operating frequency ω, we are at DC here!

Let's go back now to AC. We then have

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{E} + q\mathbf{v}\times\mathbf{B} - (m/\tau)\,\mathbf{v} \; . \qquad (N.1.5)$$

or

$$mj\omega\mathbf{v} = q\mathbf{E} + q\mathbf{v}\times\mathbf{B} - (m/\tau)\,\mathbf{v}$$

$$m(1/\tau + j\omega)\,\mathbf{v} = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}$$

$$(m/\tau)\,(1 + j\omega\tau)\,\mathbf{v} = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}$$

$$(m/q\tau)\,(1 + j\omega\tau)\,\mathbf{v} = \mathbf{E} + \mathbf{v}\times\mathbf{B}$$

$$\kappa(1 + j\omega\tau)\,\mathbf{v} = \mathbf{E} + \mathbf{v}\times\mathbf{B}$$

So this is the same as the static case but we have to take

$$\kappa \rightarrow (1+j\omega\tau)\,\kappa$$

But then we make our "frequency argument" that $\omega\tau << 1$ for our interests. Then we can use the DC results for everything regarding "magnetic Ohm's Law"! So you need both this AND the following condition.

Here now is a condition I developed for when regular Ohm's Law is OK :

Now back to the small B_i field idea! We need:

$B_i << (m/q\tau)$ ie $B_i << \kappa$ ie $\mu B_i << 1$

in order to assume regular Ohm's Law. But for copper this is just a fixed number. Here it is:

add-on to fine value of kappa

```
me := 9.109e-31* kg;
```

$$me := .9109\ 10^{-30}\ kg$$

```
tau := 1e-14*sec;
```

$$\tau := .1\ 10^{-13}\ sec$$

```
kg := cou*volt*sec^2*m^2;
```

$$kg := cou\ volt\ sec^2\ m^2$$

```
kappa := me/(q*tau);
```

$$\kappa := -569.3125000\ volt\ sec\ m^2$$

Thus, as long as B_i << 569 Tesla's, we are OK on regular Ohm's Law!!! A huge MRI machine is only about 3 Tesla.

Examples [edit]

Main article: Orders of magnitude (magnetic field)

- 31.869 μT (3.1×10^{-5} T) – strength of Earth's magnetic field at 0° latitude, 0° longitude
- 5 mT – the strength of a typical refrigerator magnet
- 0.3 T – the strength of solar sunspots
- 1.25 T – magnetic field intensity at the surface of a neodymium magnet
- 1 T to 2.4 T – coil gap of a typical loudspeaker magnet
- 1.5 T to 3 T – strength of medical magnetic resonance imaging systems in practice, experimentally up to 17 T[9]
- 4 T – strength of the superconducting magnet built around the CMS detector at CERN[10]
- 8 T – the strength of LHC magnets.
- 11.75 T – the strength of INUMAC magnets, largest MRI scanner.[11]
- 13 T – strength of the superconducting ITER magnet system [12]
- 16 T – magnetic field strength required to levitate a frog,[13] according to the 2000 Ig Nobel Prize in Physics.[14]

So here are some conclusions:

(1) Ohm's Law in the presence of a general magnetic field B is quite complicated. For example,

```
J1 := kappa*sigma*V1;
```

$$J1 := \frac{\sigma\,(\kappa\,B3\,E2 + \kappa^2\,E1 + B1^2\,E1 + B1\,B3\,E3 + B1\,E2\,B2 - \kappa\,E3\,B2)}{B1^2 + B3^2 + \kappa^2 + B2^2}$$

```
J2 := kappa*sigma*V2;
```

$$J2 := \frac{\sigma\,(B1\,\kappa\,E3 + B1\,B2\,E1 - B3\,\kappa\,E1 + B3\,E3\,B2 + E2\,\kappa^2 + E2\,B2^2)}{B1^2 + B3^2 + \kappa^2 + B2^2}$$

```
J3 := kappa*sigma*V3;
```

$$J3 := \frac{\sigma\,(B3\,B2\,E2 + E1\,B2\,\kappa + \kappa^2\,E3 - \kappa\,B1\,E2 + E3\,B3^2 + B1\,B3\,E1)}{B1^2 + B3^2 + \kappa^2 + B2^2}$$

If we replace the B_i fields in favor of their cyclotron frequencies, $B_i = \kappa\,(\omega_i\tau)$, we then get

```
J1: simplify(%);
```

$$\frac{\sigma\,(\omega3\,\tau\,E2 - \omega2\,\tau\,E3 + \omega2\,\tau^2\,\omega1\,E2 + \omega3\,\tau^2\,E3\,\omega1 + E1 + E1\,\omega1^2\,\tau^2)}{\omega3^2\,\tau^2 + \omega2^2\,\tau^2 + 1 + \omega1^2\,\tau^2}$$

```
J2: simplify(%);
```

$$\frac{\sigma\,(-\omega3\,\tau\,E1 + E2 + \omega2^2\,\tau^2\,E2 + \omega2\,\tau^2\,\omega3\,E3 + \omega2\,\tau^2\,E1\,\omega1 + E3\,\omega1\,\tau)}{\omega3^2\,\tau^2 + \omega2^2\,\tau^2 + 1 + \omega1^2\,\tau^2}$$

```
J3: simplify(%);
```

$$-\frac{\sigma\,(-\omega3\,\tau^2\,\omega1\,E1 + E2\,\omega1\,\tau - E3 - \omega2\,\tau\,E1 - E3\,\omega3^2\,\tau^2 - \omega2\,\tau^2\,\omega3\,E2)}{\omega3^2\,\tau^2 + \omega2^2\,\tau^2 + 1 + \omega1^2\,\tau^2}$$

For example, this says

$$J_1 = \sigma\,\frac{[\,E_3\{1 + (\omega_3\tau)^3\} + E_1\{(\omega_2\tau) + (\omega_1\tau)(\omega_3\tau)\} + E_2\ \{-(\omega_1\tau) + (\omega_2\tau)(\omega_3\tau)\}\,]}{1 + (\omega_1\tau)^2 + (\omega_2\tau)^2 + (\omega_3\tau)^2}$$

(2) If we then assume that $(\omega_i\tau) << 1$ for each component of the B field, and that none of the E_i are absurdly large, we end up with "regular Ohm's Law" which is this,

```
J1: simplify(%);
```

$$\sigma\,E1$$

```
J2: simplify(%);
```

$$\sigma\,E2$$

```
J3: simplify(%);
```

$$\sigma\,E3$$

which is to say, $\mathbf{J} = \sigma\mathbf{E}$.

(3) The condition that $(\omega_i\tau) << 1$ is the same as $B_i << \kappa = (m/q\tau) = 1/\mu$. For copper we find that this conditions says

B_i << 569 Tesla

The largest B fields in general use for anything on earth might be 10 T, and a transmission line B field will be much smaller than that, so this condition is met in any practical situation.

(4) Therefore, regular Ohm's Law may be used in any DC situation (with isotropic media, etc etc).

(5) What about AC? The equation of motion is adjusted in this case such that $\kappa \rightarrow (1+j\omega\tau)\,\kappa$. We can then use our DC result as long as $\omega\tau << 1$ which roughly means $\omega << 10^{13}$ Hz = 10,000 GHz. So again, for practical systems operating at AC frequencies of interest, regular Ohm's Law can be assumed!

(6) When we are studying Hall effect situations, we must use the exact Magnetic Ohm's Law in order to obtain the Hall fields.