Summary of Hall Effect in a Round Wire PhL 3.24.14

Reviewed this doc in detail on 3/31/14, I think all is resolved in "Appendix N".

We assume azimuthal symmetry and an infinitely long wire so this is a 2D problem.

Here are the governing equations:

$J_z(r) = n(r)\, e\, v_z(r)$	// current equation	$e<0$	(1)
$E_r(r) = v_z(r)\, B_\theta(r)$	// Hall effect equation		(2)
$r^{-1}\partial_r(rE_r(r)) = \rho(r)/\varepsilon_0$	// divE=0 equation		(3)
$-\partial_r E_z(r) = 0$	// curlE = 0 equation		(4)

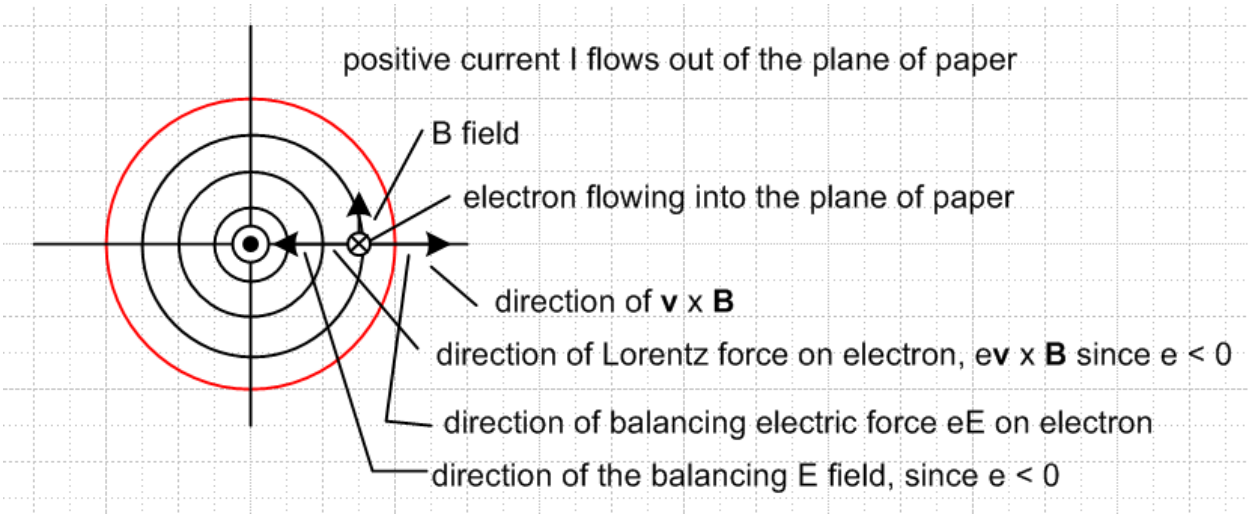
Paradox Right off the Bat. If v_z = constant, we know $B_\theta(a) \neq 0$ and then from (2) $E_r(a) \neq 0$. At least for small B this means $J_r(a) \neq 0$ [wrong because of Magnetic Ohm's Law!] and this is a contradiction because this current has nowhere to go (vacuum outside the wire). So something is wrong with equation (2). [no, but something is wrong with Ohm's Law!]

Comments:

1. In any differential operator, we can set $\partial_z[...] = 0$ and $\partial_\theta[...] = 0$ and $E_\theta = 0$. The first since everything is uniform in z, the second because nothing changes with θ, and the third due an expected symmetric solution for the E field. Certainly $E_\theta = 0$ on the surface since it is an equipotential. With these assumptions we find that div $\mathbf{E} = r^{-1}\partial_r(rE_r)$ and curl $\mathbf{E} = -\partial_r E_z \hat{\boldsymbol{\theta}}$ from which (3) and (4) above derive. [correct]

2. In the $J_z = nev_z$ equation, n is the density of electrons in the classical electron cloud inside the conductor, $e < 0$ is the electron's charge, and v_z is the average drift velocity of the flowing electrons. If we think of $J_z > 0$ flowing out of the plane of paper (the z direction), then electrons flow into the plane of paper and $v_z < 0$. Of course n(r) is positive. [correct]

3. This same v_z appears in the Hall effect equation which says there must be a local **E** field to offset the Lorentz force e **v** x **B** acting on an electrons. Consider this picture,



The **B** field is CCW by the right hand rule. We show one electron heading into paper at the cross, which indicates $v_z < 0$. The direction of **v** x **B** is radial outward. The comments in the picture are self explanatory, and we end up with a radial E field pointing inward to balance the Lorentz force. We shall find below that $E_r(r) = - K (r/a)$ where K is a positive constant, and so the E field does in fact point radially inward. Notice in the equation (2) that $E_r(r) = v_z(r) B_\theta(r) < 0$ since $B_\theta(r) > 0$ and $v_z < 0$, so we have the sign right in equation (2) as well as in equation (1). [correct]

4. None of the functions can depend on z or θ due to our geometry. We allow all functions then to possibly vary in r, even though they may end up being independent of r. [ok]

5. Normally one thinks of the free charge ρ(r) being 0 inside a conductor because the electrons offset the positive ion charges, but we allow that in this situation ρ(r) might not be 0. In fact, we know that

$$\rho(r) = e\,[n(r)-n_0] \qquad (5)$$

where n_0 is the number density of electrons in the absence of current I in the wire. [ok]

We now restate the above four equations: [all correct, note missing Ohm's Law]

$J_z(r) = n(r)\, e\, v_z(r)$ // current equation e<0 (1)

$E_r(r) = v_z(r)\, B_\theta(r)$ // Hall effect equation (2)

$r^{-1}\partial_r(rE_r(r)) = \rho(r)/\varepsilon_0$ // divE=0 equation (3)

$-\partial_r E_z(r) = 0$ // curlE = 0 equation (4)

The last equation says that $E_z(r)$ = constant. Since $J_z(r) = \sigma E_z(r)$ [an assumption, but turns out to be correct], this says J_z is a constant. [correct] This is consistent with our conclusions of Chapter 2 in the

low-frequency limit. We no longer need the last equation [correct], and our equation set then becomes, using (5) for ρ(r),

$J_z = n(r)\, e\, v_z(r)$ // current equation e<0 (6)

$E_r(r) = v_z(r)\, B_\theta(r)$ // Hall effect equation (7)

$r^{-1}\partial_r(rE_r(r)) = (e/\varepsilon_0)\,[n(r)-n_0]$ // divE=0 equation (8)

We still have several functions to worry about: n(r), $v_z(r)$ and $E_r(r)$. At least we know $B_\theta(r)$ as follows using Ampere's law with our uniform J_z : [correct]

$2\pi r\, H_\theta(r) = I(r^2/a^2) \Rightarrow \quad H_\theta(r) = [I/(2\pi a^2)]\, r \quad \Rightarrow \quad B_\theta(r) = [\mu_0 I/(2\pi a^2)]\, r = C\, r$

$C = [\mu_0 I/(2\pi a^2)]$ (9)

so $B_\theta(r)$ is a simple linear function inside the conductor. [correct]

At this point, we replace function n(r) by another function η(r) as follows

$\eta(r) = [n(r)-n_0]/n_0 = n(r)/n_0 - 1 \quad \Rightarrow \quad n(r) = n_0\,(\eta(r)+1)$ (10)

[Comment: Suppose $n(r) - n_0 = 1$ electron. Then $\eta = 1/n_0 = 1/[8.5 \times 10^{28}] = 1.1 \times 10^{-29}$.]

This dimensionless function η(r) is then the fractional deviation of n(r) from its at-rest value of n_0 . We can then say

$\rho(r) = e\,[n(r)-n_0] = e\, n_0\, \eta(r)$ [ok] (11)

Our three equations are now

$J_z = n_0\,(\eta(r)+1)\, e\, v_z(r)$ // current equation e<0 (12)

$E_r(r) = v_z(r)\, B_\theta(r)$ // Hall effect equation (13)

$r^{-1}\partial_r(rE_r(r)) = (e/\varepsilon_0)\, n_0\, \eta(r)$ // divE=0 equation [ok] (14)

and our unknown functions are $v_z(r)$, $E_r(r)$ and η(r) . We can solve (12) for $v_z(r)$ to get

$v_z(r) = J_z/(e\, n_0) * 1/(\eta(r)+1)$ (15)

and then insert this into (13) to get

$E_r(r) = J_z/(e\, n_0) * 1/(\eta(r)+1) * Cr = (J_z C/e n_0)\, \dfrac{r}{1+\eta(r)}$ (16)

and so

$$r\,E_r(r) = (J_z C/e n_0)\,\frac{r^2}{1+\eta(r)}$$

We then compute,

$$\partial_r\,\frac{r^2}{1+\eta(r)} = \frac{(1+\eta)*2r - r^2\,\eta'}{(1+\eta)^2} = \frac{2r}{1+\eta} - \frac{1}{(1+\eta)^2}\,r^2\,\eta'(r)$$

and therefore

$$\partial_r(rE_r(r)) = (J_z C/e n_0)\,[\frac{2r}{1+\eta} - \frac{1}{(1+\eta)^2}\,r^2\,\eta'(r)]$$

But according to (14) we have

$$\partial_r(rE_r(r)) = (e/\varepsilon_0)\,n_0\,r\,\eta(r)$$

Setting these last two expressions equal and dividing by r we get

$$(J_z C/e n_0)\,[\frac{2}{1+\eta} - \frac{1}{(1+\eta)^2}\,r\,\eta'(r)] = (e/\varepsilon_0)\,n_0\,r\,\eta(r)$$

or

$$[\frac{2}{1+\eta} - \frac{1}{(1+\eta)^2}\,r\,\eta'(r)] = \frac{e^2 n_0^2}{J_z C \varepsilon_0}\;\eta(r)$$

We define this constant to be β so we then have

$$[\frac{2}{1+\eta} - \frac{1}{(1+\eta)^2}\,r\,\eta'(r)] = \beta\;\eta(r) \qquad \beta = \frac{e^2 n_0^2}{J_z C \varepsilon_0}$$

Notice that β is dimensionless since all terms in the ODE here are dimensionless.

[I think all the above math regarding η and the final ODE is correct, have not checked details]

But we know that $C = [\mu_0 I/(2\pi a^2)]$ and $J_z = I/(\pi a^2)$ so

$$\beta = \frac{e^2 n_0^2}{J_z C \varepsilon_0} * \frac{J_z\,\pi a^2}{I} * \frac{C 2\pi a^2}{\mu_0 I} = \frac{e^2 n_0^2}{\varepsilon_0} * \frac{\pi a^2}{I} * \frac{2\pi a^2}{\mu_0 I} = \frac{2\pi^2 e^2 a^4 n_0^2}{\varepsilon_0 \mu_0 I^2}$$

But $1/(\varepsilon_0\mu_0) = c^2$ (c = speed of light) so then

$$\beta = \frac{2\pi^2 e^2 a^4 n_0^2 c^2}{I^2} = 2\pi^2\,[\,e a^2 n_0 c/\,I\,]^2$$

This number β is quite large for normal parameters. For example, consider,

```
I := 1;            # amps
```

$$I := 1$$

```
e := -1.602e-19;   # electron charge
```

$$e := -.1602\ 10^{-18}$$

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c := 3e8;          # light in m/sec
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$$c := .3\ 10^{9}$$

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a := 1e-2;         # 1 cm radius wire
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$$a := .01$$

```
n0 := 8.5e28;      # value for copper
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$$n0 := .85\ 10^{29}$$

```
beta := evalf(2*Pi^2*(e*a^2*n0*c/I)^2);
```

$$\beta := .3294087458\ 10^{31}$$

So we then have this ODE

$$[\frac{2}{1+\eta} - \frac{1}{(1+\eta)^2} r\, \eta'(r)] = \beta\ \eta(r) \qquad \beta \sim 10^{31}$$

We multiply through by $(1+\eta)^2$ to get

$$[2(1+\eta) - r\, \eta'] = \beta\ \eta(1+\eta)^2$$

or

$$r\, \eta' + \beta\ \eta(1+\eta)^2 - 2(1+\eta) = 0$$

or

$$r\, \eta' + (1+\eta)\ [\ \beta\eta(1+\eta) - 2] = 0$$

This is a standard issue non-linear ODE for the function η(r) with variable r. Because β is so large, it must be approximately true that

$$[\ \beta\eta(1+\eta) - 2] \approx 0$$

or

$$\eta(1+\eta) \approx 2/\beta \quad = \text{exceedingly small} \quad = \eta + \eta^2$$

Therefore η itself must be very small, and certainly $\eta << 1$. We can then approximate

$$1+\eta = 1$$

and then our ODE becomes

$$r\, \eta' + [\ \beta\eta - 2] = 0$$

which has the following solution

$$\eta(r) = C_1\, r^{-\beta} + 2/\beta$$

In order to avoid an astronomical singularity at r = 0, we must set $C_1 = 0$ and then

$$\eta(r) = 2/\beta \ll 1$$ [I think this is correct, and $\eta = 10^{-21}$ in the last App N example]

which is that same as $\eta(1+\eta) \approx 2/\beta$ if $\eta \ll 1$. We conclude then that $\eta(r)$ is a very small constant.

Recall now (15) and (16),

$$v_z(r) = J_z/(e\, n_0) * 1/(\eta(r)+1) \qquad (15)$$

$$E_r(r) = J_z/(e\, n_0) * 1/(\eta(r)+1) * Cr = (J_z C/en_0)\frac{r}{1+\eta(r)} \qquad (16)$$

We then see that $v_z(r)$ is a constant we will call v_z and

$$v_z = J_z/(e\, n_0)$$

$$E_r(r) = (J_z C/en_0)\, r = (J_z Ca/en_0)\,(r/a) = K\,(r/a)$$ [K now called - E_s]

But recall that $C = [\mu_0 I/(2\pi a^2)]$ and $J_z = I/(\pi a^2)$ so so

$$(J_z Ca/en_0) = \frac{J_z Ca}{en_0} * \frac{I}{\pi a^2 J_z} * \frac{\mu_0 I}{2\pi a^2 C} = \frac{\mu_0 I^2}{2\pi^2 a^3 e\, n_0} \equiv K$$ [K = -E_s see App N (N.7.13)]

A few extra lines of Maple,

```
mu0 := 4*Pi*1e-7;
```

$$\mu 0 := .4\ 10^{-6}\ \pi$$

```
K := mu0*I^2/(2*Pi^2*a^3*e*n0):evalf(%);
```

$$-.4675183757\ 10^{-11}$$

.

So we can write

$$E_r(r) = E_r(a)\,(r/a) \qquad E_r(a) = -\ 0.47 \times 10^{-11}\ \text{volts/m} \qquad I = 1\ \text{amp},\ \ a = 1\ \text{cm}$$

For a 1 mm radius wire, the field will be 10^3 times larger so

$$E_r(r) = E_r(a)\,(r/a) \qquad E_r(a) = -\ 0.47 \times 10^{-8}\ \text{volts/m} \qquad I = 1\ \text{amp},\ \ a = 1\ \text{mm}$$

Meanwhile we have

$v_z = J_z /(e\, n_0) = I/(\pi a^2 e n_0) = -0.23 \times 10^{-6}$ m/sec I = 1 amp, a = 1 cm

$= -0.23 \times 10^{-4}$ m/sec I = 1 amp, a = 1 mm

The charge density $\rho(r)$ is given by

$$\rho(r) = e\, n_0\, \eta(r)$$

so it is also a constant,

$$\rho = e\, n_0\, \eta = e\, n_0\, 2/\beta = -0.83 \times 10^{-20} \text{ coulombs/m}^3$$

The total charge inside a slice of thickness dz of the wire is then

$$Q = dz * \pi a^2\, \rho = dz\, \pi a^2\, e\, n_0\, 2/\beta$$

An equal and opposite charge forms on the surface of the wire. Outside the wire these charges cancel and there is no external electric field.

Conclusions

(1) the basic idea of J_z = constant and v_z = constant in a round wire carrying current I is born out. ok

(2) there is a small radial inward-directed "Hall effect" electric field inside the wire which is required to offset the Lorentz force acting on the electrons . It has the form $E_r(r) = K\,(r/a)$. ok

(3) inside the wire there is a small and constant amount of negative free charge ρ. This is what deflects the electrons curving toward the center line. ok

(4) the surface of the wire then has a small positive surface charge. ok

(5) outside the wire there is no radial electric field due to this Hall effect. ok

(6) it is not very clear how one might measure this internal electric Hall field. ok

Question: How do these facts relate to

$$r^{-1}\partial_r(rE_r(r)) = \rho(r)/\varepsilon_0 \qquad \text{// divE=0 equation} \qquad (3)$$

Answer: $E_r(r) = K\,(r/a)$ so

$$r^{-1}\partial_r(rE_r(r)) = (K/a)\, r^{-1}\partial_r(r^2) = (2K/a) = \rho/\varepsilon_0$$ OK d^2x'

Does this radial E field create some kind of measurable magnetic field outside the wire? I could use my 2D Biot-Savart to answer that question [wrong wrong! Outside field cancels and is 0.]

$$\mathbf{H}(x,y) = \int d^2x' \, \frac{1}{2\pi R^2} \, \mathbf{J}(\mathbf{x}') \times \mathbf{R} \qquad \mathbf{R} \equiv \mathbf{x} - \mathbf{x}' \qquad \text{(B.2.24)}$$

If you put point (x,y) on the right side x axis and consider contributions to this integral from two mirror points about the x axis, you find that the two contributions are equal, but of opposite sign since the cross product has opposite sign for the two pieces. Thus these two pieces contribute 0, and that applies to all pieces, so there is no external **H** field created by the Hall current.

Question: If there is a radial E_r field inside the conductor, isn't there a radial current J_r, and if there is, doesn't that violate the assumption of no transverse currents? [resolved in Magnetic Ohm's Law!]

Answer #1: Ohm's Law has the form

$$\mathbf{J} = \sigma\mathbf{E} - D \text{ grad } \rho$$

But you would think that grad $\rho = 0$ if ρ is a constant, and then we are back to $\mathbf{J} = \sigma\mathbf{E}$, so what gives here? A fly in the ointment! I will have to ponder this later.

Answer #2: OK, this is resolved now in Section 8 of "Ohms Law and Magnetic Fields".