Superpositions Practice PhL 4.1.14

[Reviewed 5/7/14] After reviewing the famous electrostatic superposition cases which involve having constant potentials on surfaces (in one case superposition fails!) , I switch to some magnetic superposition cases. In these cases the potential A_z is not mentioned and is certainly not constant on the conductor boundaries. I look at three different superposition problems and I show (I think) that they are all valid superpositions, where valid means that if Problems 1 and 2 satisfy all Maxwell equations, then so does Problem 3 = 1+2. BC's don't get much mention, and I conclude that all three cases are valid superpositions. The superposition cases are these:

(a) DC wire with current + external DC B field =
DC current with wire in external DC B field

The conclusion here is the there is no DC proximity effect and the wire current is unaffected.

(b) DC wire with current + external AC B field =
DC current with wire in external AC B field

The wire current IS affected here, so we have an AC proximity effect.

(c) AC wire with current + external AC B field =
AC current with wire in external AC B field

Again the wire current IS affected here, so we have an AC proximity effect.

I think I assume the latter two superpositions are in fact used in my Appendix P on eddy currents! At the time of writing, I was still wondering if there might be a DC proximity effect, and here was another argument that there is no such thing.

1. The canonical electrostatics superposition scenario. 1

2. Magnetic superposition examples 3

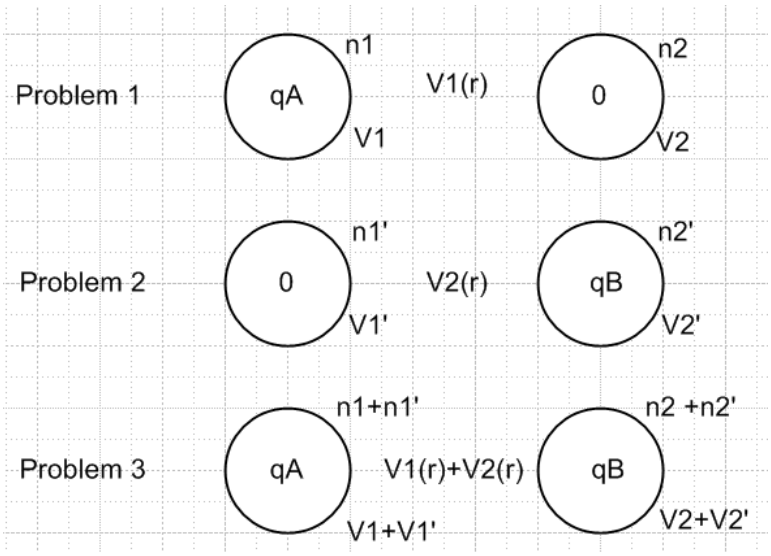
(a) Suppose everything is DC. 3

(b) Suppose Problem 1 stays DC but Problem 2 becomes AC 4

(c) Suppose both Problem 1 and Problem 2 are running at ω so both AC 6

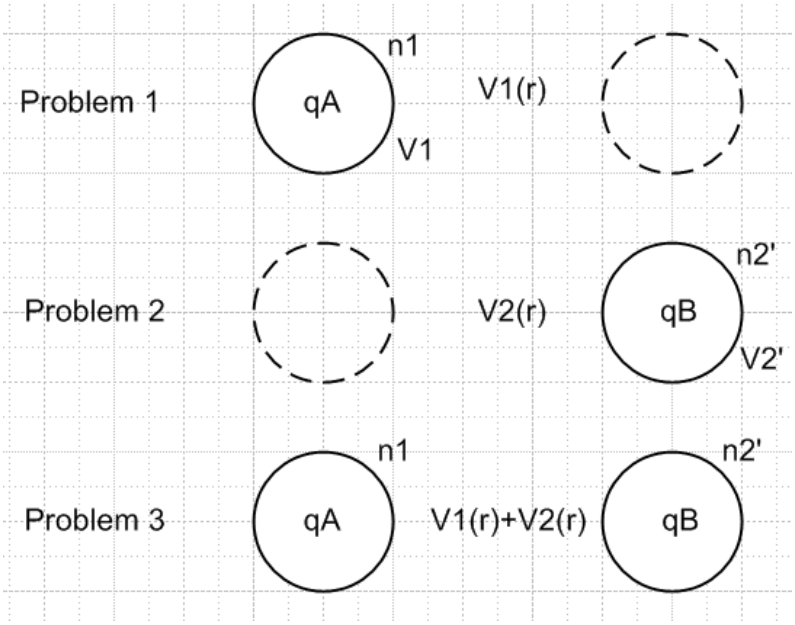
1. The canonical electrostatics superposition scenario.

In the famous electrostatic superposition of two metal spheres scenario, we have this:



In Problem 1, the right sphere is at least present, causing its surface to be an equipotential. A charge density will be induced on this sphere. Same idea for Problem 2. So is Problem 3 really a superposition here? In each problem we have the Laplace equation $\nabla^2 V = 0$ with BC's, and both the equation *and* the boundary conditions superpose correctly, so yes, it really does work. We could ground the sphere with 0 charge in problems 1 and 2 and it would still work, but we would have $V_2 = 0$ and $V_1' = 0$.

Here is what does not work:

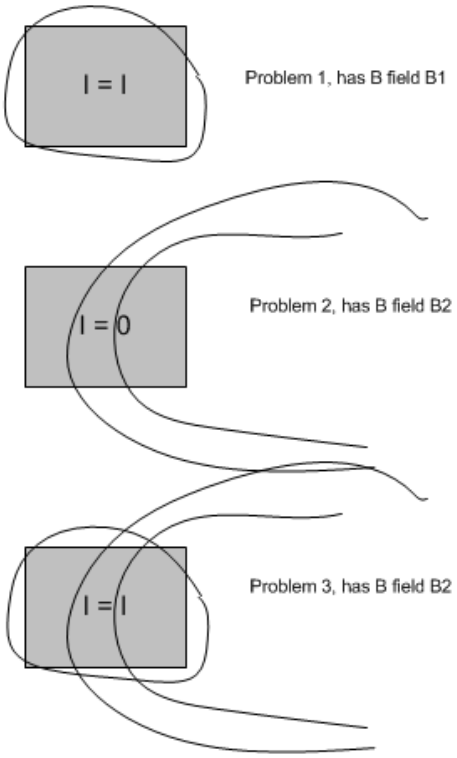


Here Problem 1 does not have $V_1(r)$ = constant on the right sphere whereas Problem 2 does, so when you add the two problems, Problem 3 does not have an equipotential surface for the right sphere so the

superposition "fails on boundary conditions". Problem 1 has no BC associated with the dotted math sphere on the right.

I recall working on this kind of question several years ago. The moral is that in each sub-problem you *have to have* all the pieces of metal hanging around that you will have in the sum problem.

2. Magnetic superposition examples



I assume that the square wire has $\mu = \mu_0$ and the vacuum is outside the wire. We shall study this situation in the following:

(a) Suppose everything is DC.

Problem 1: The conductor is in isolation and surrounded by vacuum with some $J_3 = \sigma E_3$ flowing. We assume we have examined this problem elsewhere. For this problem, presume that we have shown that v_3 and J_3 and E_3 are all constant and that there are no transverse currents and there are small transverse Hall fields and there is some tiny internal negative charge density ρ and a positive one on the surface. Current is I = J_3 x area. The current generates its own **B** field $\mathbf{B_1}$. This problem must satisfy:

curl $\mathbf{B_1} = \mu_0 \mathbf{J}$
curl $\mathbf{E} = 0$
div $\mathbf{E} = 0$ where we ignore the tiny internal negative charge density ρ auto-Hall effect.

Problem 2: The externally applied $\mathbf{B_2}$ field all by itself with the conductor just quietly sitting there. The B_2 field is generated by some external mechanism, and again it is in vacuum. This external B field is static and generates no E fields or currents in the wire and the field lines just go right through the wire. Since there is no current inside the conductor (or anywhere else) , we know that curl $\mathbf{B_2} = 0$. There is no **E** field so the other Maxwell equations are of no concern for Problem 2. Of course we must have div $\mathbf{B_2} = 0$ as well.

Each of these two problems satisfies Maxwell's equations and the Lorentz force equation.

Problem 3: We superpose the above two problems. In problem 3 we have a wire with current I, and we have a total magnetic field $\mathbf{B_1} + \mathbf{B_2}$. The E_i and J_i are exactly as in Problem 1. We then have

$$\text{curl } \mathbf{B} = \text{curl } \mathbf{B_1} + \text{curl } \mathbf{B_2} = \mu_0 \mathbf{J} + 0$$

$$\text{div } \mathbf{B} = \text{div } \mathbf{B_1} + \text{div } \mathbf{B_2} = 0 + 0 = 0$$

Thus, superposition is really a viable thing to do here.

Conclusion: The addition of the external magnetic field to a scenario of a current flowing in a wire does not alter the E_i fields in that wire and does not alter the currents J_i . If the v_z was constant without the external $\mathbf{B_2}$ field, it is still constant and in fact unchanged.

Corollary: There is no DC "proximity effect".

Now how are things different if we are doing AC? There are several possible scenarios, I will do them one at a time.

(b) Suppose Problem 1 stays DC but Problem 2 becomes AC

Problem 1 is as above, so again

$$\text{curl } \mathbf{B_1} = \mu_0 \mathbf{J_1}$$
$$\text{curl } \mathbf{E_1} = 0$$
$$\text{div } \mathbf{E_1} = 0$$
$$\text{div } \mathbf{B_1} = 0$$
$$\mathbf{J_1} = \sigma \mathbf{E_1} = \text{constant}$$

Problem 2 has an AC **B** field going at ω and the wire is present. We know that any loop inside the conductor which captures B flux will have an EMF around it, and so there is going to be some kind of current action inside the conductor and that is J_2. These are the famous eddy currents. For a given $\mathbf{B_2}$ field I could compute everything about these eddy currents $\mathbf{J_2}$, but I have never attempted that problem! We then have these equations:

$$\text{curl } \mathbf{B_2} = \mu_0 \mathbf{J_2} + j\omega\mu_0\varepsilon_0 \mathbf{E_2}$$

curl $\mathbf{E}_2 = -j\omega\mathbf{B}_2$
div $\mathbf{E}_2 = 0$ // no charge density in this problem
div $\mathbf{B}_2 = 0$

I think the $\mathbf{B}_2$ field lines just pass through the conductor unaltered except perhaps by some B field generated by this induced eddy currents which are probably small. Probably the E fields are continuous through the boundary and probably there is no surface charge, but I could easily be wrong about that. If there is surface charge, then there is some div $\mathbf{E}_2$.

Problem 3 is the superposition of the above two problems, and we look inside the conductor and we expect to have these four Maxwell equations all happy.

$\mu_0\mathbf{J}_3 = \text{curl } \mathbf{B}_3 - \partial_t\mu_0\varepsilon_0\mathbf{E}_3$ 1
curl $\mathbf{E}_3 = -\partial_t\mathbf{B}_3$ 2
div $\mathbf{E}_3 = 0$ 3
div $\mathbf{B}_3 = 0$ 4

So what do they each say? The last two will be OK obviously, but the other two I should look at:

$\mu_0\mathbf{J}_3 = \text{curl } \mathbf{B}_3 - \partial_t\mu_0\varepsilon_0\mathbf{E}_3$ 1

$\mu_0\mathbf{J}_3 = \text{curl } (\mathbf{B}_1 + \mathbf{B}_2) - \partial_t\mu_0\varepsilon_0(\mathbf{E}_1 + \mathbf{E}_2)$

$= \mu_0\mathbf{J}_1 + \mu_0\mathbf{J}_2 + j\omega\mu_0\varepsilon_0\mathbf{E}_2 - j\omega\ \mu_0\varepsilon_0\mathbf{E}_2$ // since E_1 is a constant

Therefore we find that

$\mathbf{J}_3 = \mathbf{J}_1 + \mathbf{J}_2$ 1

and so we just add the eddy currents to the Problem 1 current. Next,

curl $\mathbf{E}_3 = -\partial_t\mathbf{B}_3$

curl $(\mathbf{E}_1 + \mathbf{E}_2) = -\partial_t(\mathbf{B}_1 + \mathbf{B}_2)$

$0 - j\omega\mathbf{B}_2 = -j\omega\ \mathbf{B}_2$ // since B_1 is a constant

and this then is just an identity.

So I think we have here a successful superposition where eddy currents make there way into a DC carrying wire. This is of course how signals get into power lines and such things. So if you have just Problem 1 and you then "bring in" the external field B_2, you alter the current in the wire, and thus we for sure have some kind of **proximity effect**.

(c) Suppose both Problem 1 and Problem 2 are running at ω so both AC

Problem 1: I guess we imagine the wire as the center conductor of a coax line as in lines doc. There will be skin effect and there will be surface charge. There will be Maxwell equations:

curl $\mathbf{B}_1 = \mu_0\mathbf{J}_1 + j\omega\mu_0\varepsilon_0\mathbf{E}_1$
curl $\mathbf{E}_1 = -j\omega\mathbf{B}_1$
div $\mathbf{E}_1 = 0$ // except at surface since now there is surface charge
div $\mathbf{B}_1 = 0$

Problem 2: Exactly same as in previous case.

curl $\mathbf{B}_2 = \mu_0\mathbf{J}_2 + j\omega\mu_0\varepsilon_0\mathbf{E}_2$
curl $\mathbf{E}_2 = -j\omega\mathbf{B}_2$
div $\mathbf{E}_2 = 0$ // no charge density in this problem
div $\mathbf{B}_2 = 0$

Problem 3:

$\mu_0\mathbf{J}_3 =$ curl $\mathbf{B}_3 - \partial_t\mu_0\varepsilon_0\mathbf{E}_3$ 1
curl $\mathbf{E}_3 = -\partial_t\mathbf{B}_3$ 2
div $\mathbf{E}_3 = 0$ 3
div $\mathbf{B}_3 = 0$ 4

So equation 1 says

$\mu_0\mathbf{J}_3 =$ curl $(\mathbf{B}_1 + \mathbf{B}_2) - j\omega\mu_0\varepsilon_0(\mathbf{E}_1 + \mathbf{E}_2)$ 1

$= \text{curl } \mathbf{B}_1 + \text{curl } \mathbf{B}_2 - j\omega\mu_0\varepsilon_0\mathbf{E}_1 - j\omega\mu_0\varepsilon_0\mathbf{E}_2$

$= \mu_0\mathbf{J}_1 + j\omega\mu_0\varepsilon_0\mathbf{E}_1 + \mu_0\mathbf{J}_2 + j\omega\mu_0\varepsilon_0\mathbf{E}_2 - j\omega\mu_0\varepsilon_0\mathbf{E}_1 - j\omega\mu_0\varepsilon_0\mathbf{E}_2$

$= \mu_0\mathbf{J}_1 + \mu_0\mathbf{J}_2$

which again says

$\mathbf{J}_3 = \mathbf{J}_1 + \mathbf{J}_2$

Next we have

curl $\mathbf{E}_3 = -\partial_t\mathbf{B}_3$ 2

curl $(\mathbf{E}_1 + \mathbf{E}_2) = -j\omega\ (\mathbf{B}_1 + \mathbf{B}_2)$

But this again is just an identity, which is a good thing. Finally

div $\mathbf{E}_3$ = div $\mathbf{E}_1$ = the surface charge stuff

div $\mathbf{B}_3$= 0

So here we have the transmission line "proximity effect". The **B** field from "the other conductor" is changing at ω, and it does something to the current in our wire. I think this should agree with my partial wave expansion conclusion where we get asymmetric J_z.